Lecture Notes for College Physics I

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1 Vector Algebra

Textbook Reference: Chapter 3 – sections 1-6 & Appendix A.

- **Definition of a Vector**

○ A vector $\mathbf{v}$ is determined in terms of its magnitude $v = |\mathbf{v}|$ and its direction $\widehat{\mathsf{v}} = \mathbf{v}/v$.

○ In two-dimensional space, a vector $\mathbf{v}$ is written as

$$\mathbf{v} \;=\; v_x\,\widehat{\mathsf{x}} \;+\; v_y\,\widehat{\mathsf{y}},$$

where $\widehat{\mathsf{x}} = (1,0)$ and $\widehat{\mathsf{y}} = (0,1)$ are unit vectors in the directions of increase of x and y, respectively, and (v_x, v_y) denotes its components.

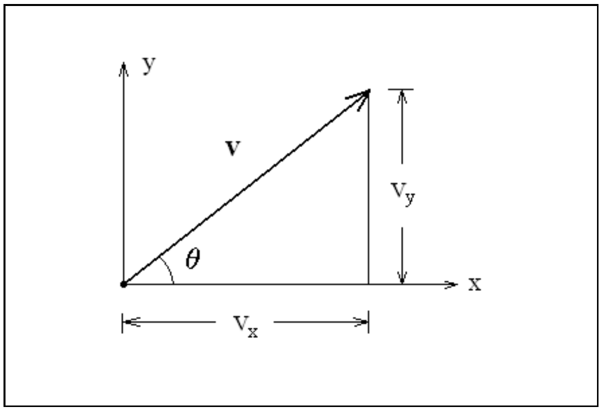
○ In terms of its components (v_x, v_y), the magnitude of the vector $\mathbf{v}$ is

$$v \;=\; |\mathbf{v}| \;=\; \sqrt{v_x^2 + v_y^2},$$

while its direction is

$$\widehat{\mathsf{v}} \;=\; \frac{v_x}{\sqrt{v_x^2 + v_y^2}}\,\widehat{\mathsf{x}} \;+\; \frac{v_y}{\sqrt{v_x^2 + v_y^2}}\,\widehat{\mathsf{y}}.$$

Note: The magnitude of a vector is always **positive**.



○ The direction unit vector $\widehat{\mathsf{v}}$ can also be represented in terms of the direction angle θ as

$$\widehat{\mathsf{v}} \;=\; \cos\theta\,\widehat{\mathsf{x}} \;+\; \sin\theta\,\widehat{\mathsf{y}}.$$

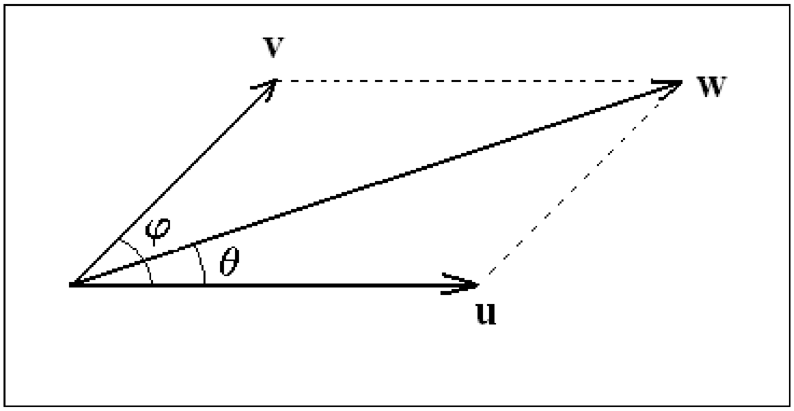
- **Vector Algebra**

◦ Multiplication of a Vector $\mathbf{v}$ by a Scalar α

$$\alpha\,\mathbf{v} \;=\; (\alpha\, v_x)\,\hat{\mathsf{x}} \;+\; (\alpha\, v_y)\,\hat{\mathsf{y}} \quad\rightarrow\quad \left\{ \begin{array}{rcl} |\alpha\,\mathbf{v}| & = & \sqrt{(\alpha\, v_x)^2 + (\alpha\, v_y)^2} = |\alpha|\;|\mathbf{v}| \\ \\ \widehat{\alpha\,\mathbf{v}} & = & (\alpha/|\alpha|)\;\hat{\mathsf{v}} \end{array} \right.$$

◦ Vector addition $\mathbf{w} = \mathbf{u} + \mathbf{v}$

$$\mathbf{w} \;=\; w_x\,\hat{\mathsf{x}} \;+\; w_y\,\hat{\mathsf{y}} \;=\; (u_x + v_x)\,\hat{\mathsf{x}} \;+\; (u_y + v_y)\,\hat{\mathsf{y}}$$



Example: $\mathbf{u} = u\,\hat{\mathsf{x}}$ and $\mathbf{v} = v\,(\cos\varphi\,\hat{\mathsf{x}} + \sin\varphi\,\hat{\mathsf{y}})$

$$w \;=\; \sqrt{u^2 + v^2 + 2\,uv\,\cos\varphi} \quad \text{and} \quad \tan\theta \;=\; \frac{v\,\sin\varphi}{u + v\,\cos\varphi}$$

Note: Direction angle θ is defined as follows

$$\theta \;=\; \left\{ \begin{array}{ll} \arctan(w_y/w_x) & \text{if} \quad w_x \;>\; 0 \\ \\ \pi + \arctan(w_y/w_x) & \text{if} \quad w_x \;<\; 0 \end{array} \right.$$

Test your knowledge: Problems 4-7 & 12 of Chapter 3

2 Kinematics of Two-Dimensional Motion

Textbook Reference: Chapter 3 – section 6.

- **Vector Kinematics**

Kinematics is the part of Physics that contains the terminology used to describe the motion of particles. For this purpose, the first element of the kinematic description of the motion of a particle involves tracking its position as a function of time. Because the motion of the particle may involve more than one spatial dimension, a vector representation is adopted. Hence, the position of a particle at time t is denoted as $\mathbf{r}(t)$ or, assuming that the motion takes place in two dimensions, as

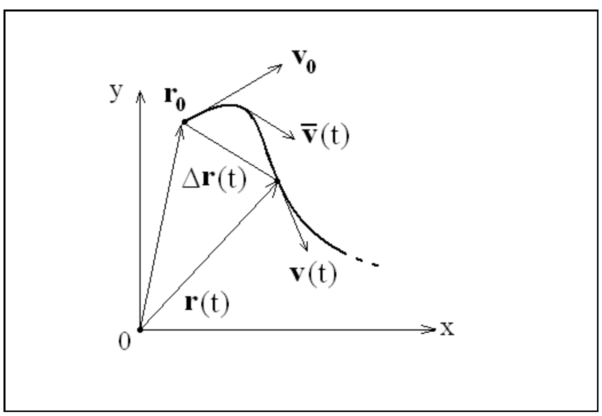
$$\mathbf{r}(t) \;=\; x(t)\,\widehat{\mathsf{x}} \;+\; y(t)\,\widehat{\mathsf{y}} \;=\; (x(t),\; y(t)).$$

Since the initial position of the particle is often known, we denote the initial position of the particle as $\mathbf{r}_0 = x_0\,\widehat{\mathsf{x}} + y_0\,\widehat{\mathsf{y}} = (x_0,\; y_0)$. Questions that immediately arise are (a) the distance covered and (b) the net displacement experienced by the particle in going from the initial position $\mathbf{r}_0$ to the instantaneous position $\mathbf{r}(t)$. It turns out that we cannot answer the first question without known the path taken by the particle between these two points (i.e., the particle may zigzag its way between the two points making the distance covered far greater than the distance separating the two points). If the path taken is a straight line, however, then the distance covered by the particle is simply given by the distance $|\mathbf{r}(t) - \mathbf{r}_0|$ between the two points

$$|\mathbf{r}(t) - \mathbf{r}_0| \;=\; \sqrt{(x(t) - x_0)^2 \;+\; (y(t) - y_0)^2}.$$

The answer to the second question introduces the definition of the net displacement vector

$$\Delta\mathbf{r}(t) \;=\; \mathbf{r}(t) \;-\; \mathbf{r}_0 \;=\; [x(t) - x_0]\,\widehat{\mathsf{x}} \;+\; [y(t) - y_0]\,\widehat{\mathsf{y}}.$$



Note that the net displacement experienced by a particle is zero if the particle returns to its initial point, while the distance covered by the particle along this closed path is not zero.

Now that we know how to describe the position $\mathbf{r}(t)$ of a particle and find its displacement $\Delta\mathbf{r}(t)$ from an initial position $\mathbf{r}_0$, we are interested in describing how fast the particle is moving. Here, we need to introduce the concepts of **instantaneous** velocity $\mathbf{v}(t)$ and

averaged velocity $\overline{\mathbf{v}}(t)$. The averaged velocity is simply calculated by determining the net dispacement $\Delta\mathbf{r}(t)$ experienced by a particle during a given time interval $\Delta t = t$ and calculating the ratio:

$$\overline{\mathbf{v}}(t) \;=\; \frac{\Delta\mathbf{r}(t)}{\Delta t} \;=\; \frac{\Delta x(t)}{\Delta t}\,\widehat{\mathsf{x}} \;+\; \frac{\Delta y(t)}{\Delta t}\,\widehat{\mathsf{y}}.$$

The instantaneous velocity $\mathbf{v}(t)$, however, can only be defined mathematically (i.e., we know it exists physically but cannot measure it experimentally!). The instantaneous velocity $\mathbf{v}(t)$ can be calculated from the averaged velocity $\overline{\mathbf{v}}(t)$ by letting the interval Δt *go to* zero:

$$\mathbf{v}(t) \;=\; \lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}(t+\Delta t)}{\Delta t} \;=\; \frac{d\mathbf{r}(t)}{dt} \;=\; \frac{dx(t)}{dt}\,\widehat{\mathsf{x}} \;+\; \frac{dy(t)}{dt}\,\widehat{\mathsf{y}},$$

which naturally introduces the concept of a time **derivative**. Note from the Figure shown above that the instantaneous velocity $\mathbf{v}(t)$ is **always tangent** to the curve $\mathbf{r}(t)$ representing the path of the particle while the averaged velocity $\overline{\mathbf{v}}(t)$ is always in the same direction as the net displacement $\Delta\mathbf{r}(t)$.

Whereas the concept of velocity is associated with displacement, the concept of **speed** is associated with the distance covered by the particle. In fact, a *speedometer* measures only speed while the determination of velocity requires the direction of motion (e.g., given by a compass) as well as its speed.

The last kinematic attribute of particle motion involves the determination of whether the instantaneous velocity $\mathbf{v}(t)$ of the particle changes as a function of time. We define the instantaneous acceleration vector $\mathbf{a}(t)$ to be the rate of change of the instantaneous velocity $\mathbf{v}(t)$:

$$\mathbf{a}(t) \;=\; \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}(t+\Delta t)}{\Delta t} \;=\; \frac{d\mathbf{v}(t)}{dt} \;=\; \frac{dv_x(t)}{dt}\,\widehat{\mathsf{x}} \;+\; \frac{dv_y(t)}{dt}\,\widehat{\mathsf{y}},$$

while the averaged acceleration is simply defined as $\overline{\mathbf{a}} = \Delta\mathbf{v}/\Delta t$. Note that the instantaneous acceleration of a particle can also be defined in terms of the second time derivative of its instantaneous position $\mathbf{r}(t)$:

$$\mathbf{a}(t) \;=\; \frac{d^2\mathbf{r}(t)}{dt^2} \;=\; \frac{d^2x(t)}{dt^2}\,\widehat{\mathsf{x}} \;+\; \frac{d^2y(t)}{dt^2}\,\widehat{\mathsf{y}}.$$

- **Motion under Constant Acceleration**

When a particle moves under constant acceleration $\mathbf{a}$, its instantaneous velocity $\mathbf{v}(t)$ is defined as

$$\mathbf{v}(t) \;=\; \mathbf{v}_0 \;+\; \mathbf{a}\,t \;=\; (v_{x0} + a_x\,t,\; v_{y0} + a_y\,t),$$

where $\mathbf{v}_0$ denotes the particle's initial velocity, while its instantaneous position $\mathbf{r}(t)$ is defined as

$$\mathbf{r}(t) \;=\; \mathbf{r}_0 \;+\; \mathbf{v}_0\,t \;+\; \frac{1}{2}\,\mathbf{a}\,t^2 \;=\; \left(x_0 + v_{x0}\,t + \frac{1}{2}\,a_x\,t^2,\; y_0 + v_{y0}\,t + \frac{1}{2}\,a_y\,t^2\right).$$

Note that from these definitions we recover $\mathbf{v}(t) = d\mathbf{r}(t)/dt$ and $\mathbf{a} = d\mathbf{v}(t)/dt = d^2\mathbf{r}(t)/dt^2$. Note also that the averaged velocity $\overline{\mathbf{v}}(t)$ obeys the equation

$$\overline{\mathbf{v}}(t) \;=\; \mathbf{v}_0 \;+\; \frac{1}{2}\,\mathbf{a}\,t,$$

i.e., the averaged velocity still evolves linearly but now at **half** the value of the constant acceleration.

Lastly, we note that the components $(x, x_0;\, v_x, v_{x0};\, a_x)$ and $(y, y_0;\, v_y, v_{y0};\, a_y)$ each independently satisfy the relation

$$v_x^2 \;=\; v_{x0}^2 \;+\; 2\,a_x\,(x - x_0) \quad \text{and} \quad v_y^2 \;=\; v_{y0}^2 \;+\; 2\,a_y\,(y - y_0),$$

where the time coordinate t has been completely eliminated.

Test your knowledge: Problems 18, 21 & 23 of Chapter 3

3 Projectile Motion

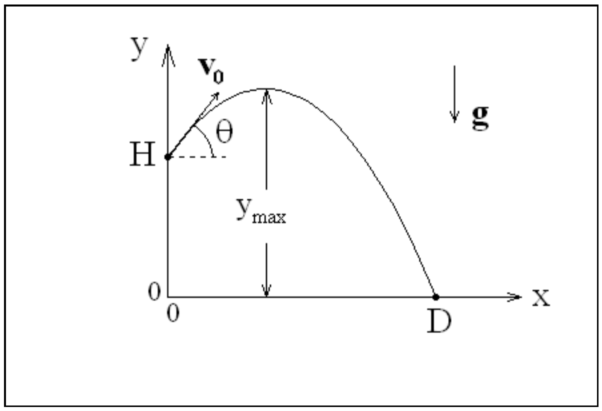
Textbook Reference: Chapter 3 – sections 7-8.

The problem of projectile motion involves the kinematic description of the path of an object (the *projectile*) moving in the presence of the constant gravitational acceleration $\mathbf{a} = -\,g\,\widehat{\mathsf{y}}$, where $g = 9.8067...$ m/s denotes the standard value of g.

The equations of projectile motion are

$$\begin{aligned} x(t) &= v_0\,\cos\theta\,t \\ y(t) &= H \;+\; v_0\,\sin\theta\,t \;-\; \frac{g}{2}\,t^2 \\ v_y(t) &= v_0\,\sin\theta \;-\; g\,t \end{aligned}$$

where H denotes the initial height at which the projectile was launched (for convenience, we set the initial horizontal position $x_0 = 0$), v_0 denotes the initial launch speed and θ denotes the initial launch angle (see Figure below).



As can be seen from the Figure above, the projectile reaches a maximum height, denoted y_{max}, as its vertical velocity v_y reaches zero (if $0 < \theta < 90^o$) as a result of the fact that the downward graviational acceleration slows down the upward motion until it stops. From that moment onward, the vertical component of the projectile's velocity becomes negative and continues to increase until the projectile hits the ground (at $y = 0$) after having covered a horizontal distance D.

- **Maximum Height**

The projectile reaches a maximum height y_{max} after a time T_{max} defined as the time when the vertical component $v_y(t)$ vanishes:

$$v_y(T_{max}) \;=\; 0 \;=\; v_0 \,\sin\theta \;-\; g\,T_{max} \quad\to\quad T_{max} \;=\; \frac{v_0\,\sin\theta}{g}.$$

Using that time, we may now calculate the vertical position of the projectile as

$$y_{max} = y(T_{max}) \;=\; H \;+\; v_0\,\sin\theta \;\cdot\; \frac{v_0\,\sin\theta}{g} \;-\; \frac{g}{2}\left(\frac{v_0\,\sin\theta}{g}\right)^2 \;=\; H \;+\; \frac{(v_0\,\sin\theta)^2}{2\,g},$$

while it has traveled the horizontal distance

$$x(T_{max}) \;=\; v_0\,\cos\theta \;\cdot\; \frac{v_0\,\sin\theta}{g} \;=\; \frac{v_0^2}{g} \;\cdot\; \cos\theta\,\sin\theta.$$

- **Free Fall**

The next phase of the projectile motion after the object starts its descent involves the projectile *falling* from a height y_{max} with zero *initial* vertical velocity $v_{y\,max} = 0$. The equation of motion for this free-fall phase can be written as

$$y(t) \;=\; y_{max} \;-\; \frac{g}{2}\,t^2 \quad\text{and}\quad v_y(t) \;=\; -\,g\,t.$$

The projectile, therefore, hits the ground after a time T_{fall} has elapsed (since the time it has reached the maximum height y_{max}), which is defined as

$$y(T_{fall}) \;=\; y_{max} \;-\; \frac{g}{2}\,T_{fall}^2 \quad\to\quad T_{fall} \;=\; \sqrt{\frac{2\,y_{max}}{g}}.$$

At the time when the projectile hits the ground, it has traveled an additional horizontal distance

$$x(T_{fall}) \;=\; v_0\,\cos\theta \;\cdot\; \sqrt{\frac{2\,y_{max}}{g}}.$$

- **Maximum Horizontal Distance**

When we add up the horizontal distances $x(T_{max})$ and $x(T_{fall})$, we can calculate the total horizontal distance covered by the projectile before it hits the ground as

$$D \;=\; v_0\,\cos\theta \,\cdot\, (T_{max} + T_{fall}) \;=\; v_0\,\cos\theta\left(\frac{v_0\,\sin\theta}{g} \;+\; \sqrt{\frac{2H}{g} \;+\; \frac{(v_0\,\sin\theta)^2}{g^2}}\right),$$

which is also involves the solution of the quadratic equation

$$y(T_{tot}) \;=\; 0 \;=\; H \;+ v_0\,\sin\theta\,T_{tot} \;-\; \frac{g}{2}\,T_{tot}^2 \quad\to\quad D \;=\; x(T_{tot}) \;=\; v_0\cos\theta\,T_{tot}.$$

- **Special Cases**

Two special cases present themselves. The first special case deals with the situation where the projectile is launched straight up or straight down (i.e., $\theta = \pm 90^o$) from some initial height H. The second special case deals with the situation where the projectile is launched horizontally (i.e., $\theta = 0$) from some initial height H.

- **Parabolic Motion**

We can eliminate the time coordinate t from the equations of projectile motion in favor of the horizontal position

$$x(t) \;=\; v_0 \cos\theta\, t \quad \rightarrow \quad t \;=\; \frac{x}{v_0 \cos\theta},$$

which can then be substituted in the equation for the vertical position

$$y(x) \;=\; H \;+\; x\;\tan\theta \;-\; \frac{g \sec^2\theta}{2\,v_0^2}\,x^2,$$

which now describes a parabola in the (x, y)-plane.

Test your knowledge: Problems 26 & 29 of Chapter 3

4 Newton's Laws of Motion

Textbook Reference: Chapter 4 – sections 1-6.

We now enter the realm of the **dynamics** of particles after having spent some time discussing the **kinematics** of particles. The issues discussed in dynamics center on the **causes** of motion, which are known as **forces**.

- **Newton's First Law of Motion**

A body continues in its state of rest or of uniform linear motion until it is acted upon by a net force.

The tendency of a body to maintain its state of rest or of uniform linear motion (i.e., constant velocity) is called **inertia**. The physical measurement of inertia is called **mass**.

- **Newton's Second Law of Motion**

> **The net acceleration of an object is directly proportional to (and in the same direction as) the net force acting on it and is inversely proportional to its mass.**

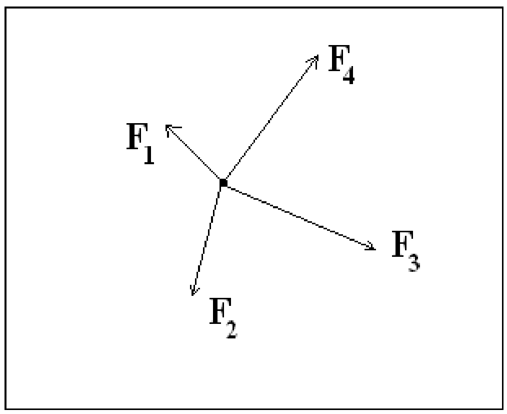
- **Newton's Third Law of Motion**

> **Whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first.**

Forces as Vectors

When several forces $(\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_N)$ are acting on an object of mass m, the net force is calculated as the vector sum

$$\begin{aligned} \mathbf{F}_{net} &= \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_N \;=\; \sum_{i=1}^{N} \mathbf{F}_i \\ &= \left(\sum_{i=1}^{N} F_{x\,i} \right) \widehat{\mathsf{x}} \;+\; \left(\sum_{i=1}^{N} F_{y\,i} \right) \widehat{\mathsf{y}} \end{aligned}$$



Newton's Second Law states that the net acceleration $\mathbf{a}_{net}$ experienced by the object is

$$\mathbf{a}_{net} \;=\; \mathbf{F}_{net}/m.$$

An object is, therefore, in a state of rest whenever the net force acting on it is $\mathbf{F}_{net} = 0$. We note that the unit of force is the *Newton* (abbreviated N) and is defined as

$$1\ \mathrm{N} \;=\; 1\,\mathrm{kg}\;\cdot\;1\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

- **Forces as Derivatives of Kinematic Quantities**

Using kinematic definitions of velocity and acceleration, we also express the force on an object of mass m (assumed to be constant) as

$$\mathbf{F} \;=\; m\,\mathbf{a} \;=\; m\,\frac{d\mathbf{v}}{dt} \;=\; m\,\frac{d^2\mathbf{x}}{dt^2}.$$

Hence, we may define the *average* force

$$\overline{\mathbf{F}} \;=\; m\,\frac{\Delta\mathbf{v}}{\Delta t}$$

in terms of the change in velocity $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$ experienced during the time interval Δt.

If the object's mass m is not constant (i.e., $dm/dt \neq 0$), the Newton's Second Law is now stated as

$$\mathbf{F} \;=\; \frac{d(m\,\mathbf{v})}{dt} \;=\; m\,\frac{d\mathbf{v}}{dt} \;+\; \frac{dm}{dt}\,\mathbf{v},$$

which forms the dynamical basis of rocket propulsion by combining Newton's Second and Third Laws.

- **Weight – The Force due to Gravity**

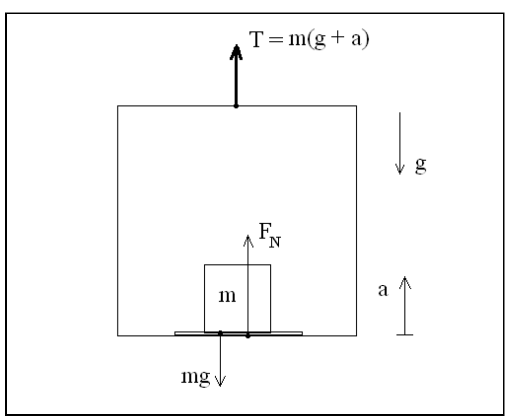
When an object of mass m is exposed to the downward (near-Earth) gravitational acceleration $\mathbf{a} = -\,g\;\widehat{\mathsf{y}}$ (where $g = 9.80$ m/s^2 denotes the magnitude of the near-Earth graviational acceleration), the object experiences a downard force $\mathbf{F}_G = -\,mg\;\widehat{\mathsf{y}}$ whose magnitude mg is known as the weight of the object.

When an object (O) of mass m is at **rest** on a horizontal surface (S), the object exerts a force $\mathbf{F}_{O\rightarrow S} = -\,mg\,\widehat{\mathsf{y}}$ on the surface while, according to Newton's First and Third Laws, the surface must exert a force

$$\mathbf{F}_{S\rightarrow O} \;=\; -\,\mathbf{F}_{O\rightarrow S} \;=\; mg\,\widehat{\mathsf{y}}$$

back on the object in the opposite direction but of the same magnitude. The *reaction* force on the object is known as the **normal** force $\mathbf{F}_N$, since this force is **always** directed perpendicular to the surface.

As an example, we consider the case of an object of mass m resting at the bottom of a massless elevator suspended by a cable with tension T.



When the elevator is at rest, the tension T in the cable is obviously equal to the weight mg of the object: $T = mg$. Because the object is resting at the bottom of the elevator, it exerts a downward force mg on the bottom floor, which reacts back with the normal force $F_N = mg$ equal to the weight of the object.

When the elevator is accelerated **upward** with net acceleration a, on the one hand, the tension in the cable must now be greater than the weight of the object: $T = m\,(g+a) > mg$. In turn, because the object is now exerting a force $m\,(g+a)$ on the bottom of the elevator greater than its weight, the normal force is now $F_N = m\,(g+a)$. This can easily be seen from the fact that, in the non-inertial frame of reference of the elevator, the weight of the object is now perceived to be $m\,(g+a)$ but since the object is at rest in that frame of reference, the normal force has to be $F_N = m\,(g+a)$. When the elevator is accelerated downward (i.e., with acceleration $-a$), on the other hand, the tension T and the normal force F_N are both less than the weight of the object and $T = m\,(g-a) = F_N$.

Hence, we find that the normal force $\mathbf{F}_N$ for this problem is equal to and opposite to the tension force $\mathbf{T}$ in the cable. In fact, by cutting the cable, we reduce the tension to zero (and, therefore, the normal force), since the acceleration is now $a = -g$; an object placed in a free-falling elevator can, thus, be considered as *weightless*.

Test your knowledge: Problems 4, 6 & 9 of Chapter 4

5 Force Problems

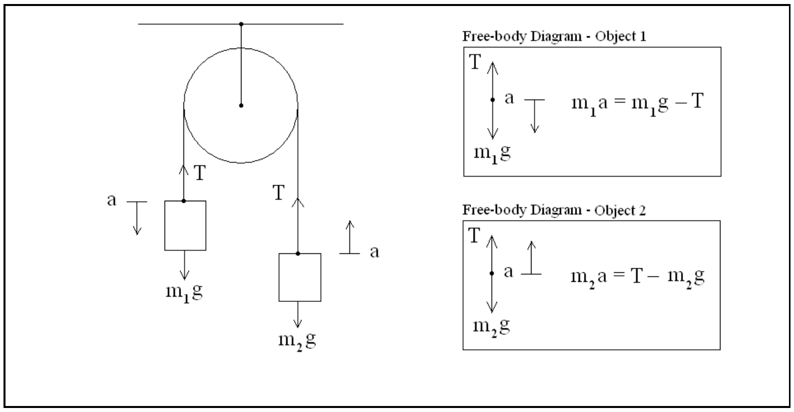
Textbook Reference: Chapter 4 – sections 7-8.

Force problems typically require the calculation of the net acceleration from applications of Newton's Three Laws of Motion. Such problems are solved by following a four-step method that focuses on **free-body force** diagrams.

I. Draw a sketch of the problem.
II. For each object, draw a free-body force diagram showing only the forces acting directly on the object.
III. For each object, write down all components of Newton's Second Law relating the system acceleration to the net force acting on the object.
IV. Solve the equation or coupled equations for the unknown(s).

- **Atwood Machine**

As our first example, we consider the simple Atwood machine (see below) composed of a *massless* pulley and two objects of mass m_1 and m_2 connected together through a *massless* string.



Assuming that $m_2 > m_1$, our intuition tells us that the system will acquire a net acceleration a directed as shown in the Figure above (Step I). The questions associated with the Atwood machine are (a) determine the system acceleration a and (b) determine the tension T in the string.

In Step II, we draw free-body force diagrams showing only the forces acting directly on mass m_1 and m_2, respectively (see above). Next, in Step III, for each free-body force diagram, we write down Newton's Second Law and, for the Atwood machine, we find

$$m_1\,a \;=\; T \;-\; m_1\,g, \tag{1}$$

$$m_2\,a \;=\; m_2\,g \;-\; T. \tag{2}$$

Since this is a set of two equations for two unknowns (a and T), we can find a unique solution for the system acceleration a and the string tension T (Step IV). Note that by adding the two equations (1) and (2), the tension T drops out, yielding the equation $(m_1 + m_2)\,a = (m_2 - m_1)\,g$, from which we obtain the system acceleration

$$a \;=\; \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g. \tag{3}$$

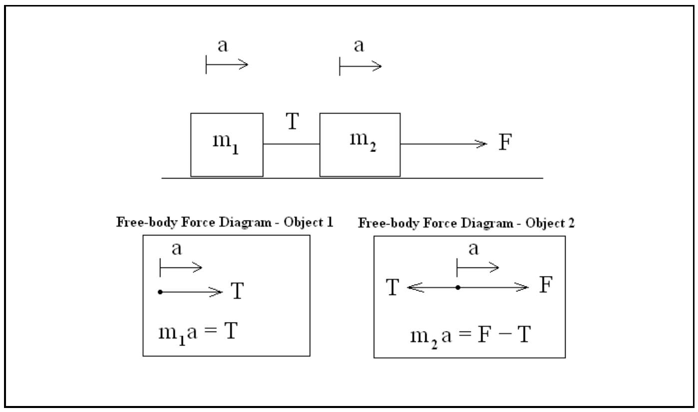
Next, we substitute this expression into Eq. (1) or (2) to find the string tension

$$\begin{aligned} T &= m_1\,(g + a) \;=\; \left(m_1 \;+\; \frac{m_1\,(m_2 - m_1)}{(m_2 + m_1)}\right) g \\ &= \left(\frac{m_1\,(m_2 + m_1) \;+\; m_1\,(m_2 - m_1)}{(m_2 + m_1)}\right) g \\ &= \left(\frac{2\,m_1 m_2}{m_1 + m_2}\right)\; g. \end{aligned} \tag{4}$$

Note that if $m_1 > m_2$, then the sign of the acceleration (3) changes (i.e., the motion changes direction) while the tension (4) stays the same.

- **Pulling on Boxes**

For our second problem, we consider applying a force F on two boxes tied together by a massless string.



By proceeding with the free-body force method, we quickly arrive at the coupled equations

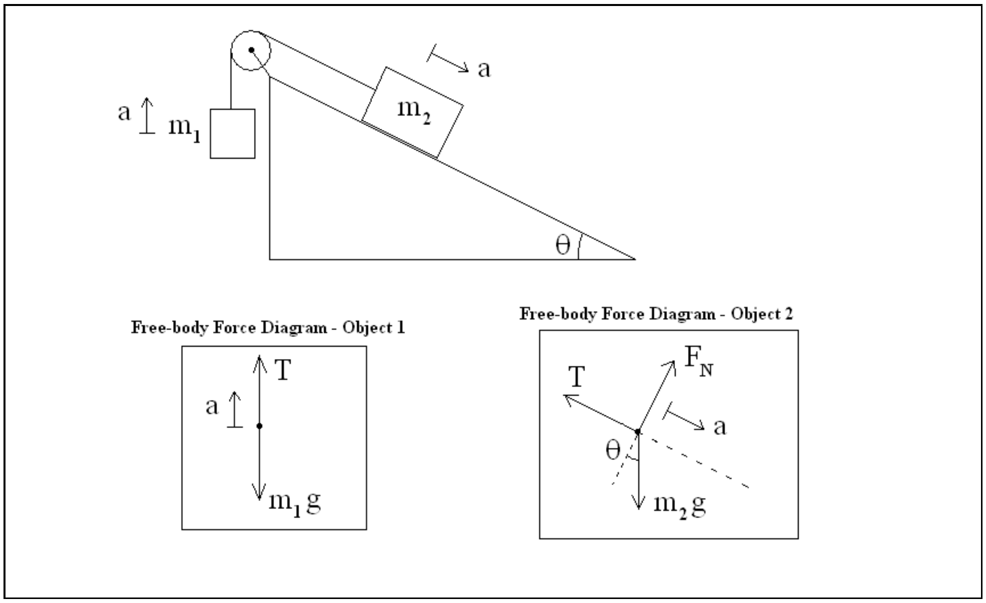
$$m_1\,a \;=\; T, \tag{5}$$
$$m_2\,a \;=\; F \;-\; T \tag{6}$$

for the unknown system acceleration a and string tension T, which are then solved respectively as

$$a \;=\; \frac{F}{m_1 + m_2} \quad \text{and} \quad T \;=\; m_1\,a \;=\; \left(\frac{m_1}{m_1+m_2}\right) F \;<\; F. \tag{7}$$

- **Frictionless Inclined Plane**

Our last problem for this lecture involves the combination of a simple Atwood machine with motion on an inclined planed (which makes an angle θ with respect to the horizontal plane). Here, we assume that the object of mass m_2 is sliding down the frictionless inclined planed with acceleration a and, consequently, the object of mass $m_1 < m_2$ is being pulled upward with the same acceleration (see Figure below).



From the free-body force diagram for object 1, we find Newton's Second Law

$$m_1\, a \;=\; T \;-\; m_1\, g, \tag{8}$$

where the system acceleration a and the string tension T are unknowns. From the free-body force diagram for object 2, we find the following parallel and perpendicular components of Newton's Second Law

$$\left.\begin{array}{rcl} m_2\, a & = & m_2 g\, \sin\theta \;-\; T \\ \\ F_N & = & m_2 g\, \cos\theta \end{array}\right\}, \tag{9}$$

where F_N denotes the normal force provided by the inclined plane. Note that since object 2 is NOT accelerated off the inclined plane, the normal force F_N must be equal to the component $m_2 g\, \cos\theta$ of the weight of object 2 perpendicular to the inclined plane.

Once again, we solve for the system acceleration a by adding Eq. (8) with the first equation in Eq. (9) to obtain

$$(m_1 + m_2)\, a \;=\; (m_2\, \sin\theta - m_1)\; g \quad \rightarrow \quad a \;=\; \left(\frac{m_2\, \sin\theta - m_1}{m_2 + m_1} \right) g. \tag{10}$$

The string tension T, on the other hand, is solved from Eqs. (8) and (10)

$$\begin{aligned} T \;&=\; m_1\,(g + a) \;=\; \left(m_1 \;+\; \frac{m_1\,(m_2 \sin\theta - m_1)}{(m_2 + m_1)} \right) g \\ &=\; \left(\frac{m_1\,(m_2 + m_1) \;+\; m_1\,(m_2 \sin\theta - m_1)}{(m_2 + m_1)} \right) g \\ &=\; \left(\frac{m_1 m_2}{m_1 + m_2} \right)\; g\,(1 + \sin\theta). \end{aligned} \tag{11}$$

Note that the simple Atwood machine becomes a special case of this problem associated with $\theta = \frac{\pi}{2}$. In addition, if $m_1 > m_2 \sin\theta$, then the motion is reversed, with a new system acceleration

$$a \;=\; \left(\frac{m_1 - m_2 \sin\theta}{m_2 + m_1} \right) g \quad (\text{if } m_1 > m_2 \sin\theta).$$

Test your knowledge: Problems 26 & 31 of Chapter 4

6 Forces due to Friction and Uniform Circular Motion

Textbook Reference: Chapter 5 – sections 1-3.

Until now, we have considered only three different types of forces: forces due to gravity, normal forces provided by supporting surfaces, and tension forces. In the present Lecture Notes, we investigate new types of force: forces due to friction and forces due to uniform circular motion.

- **Forces due to Friction**

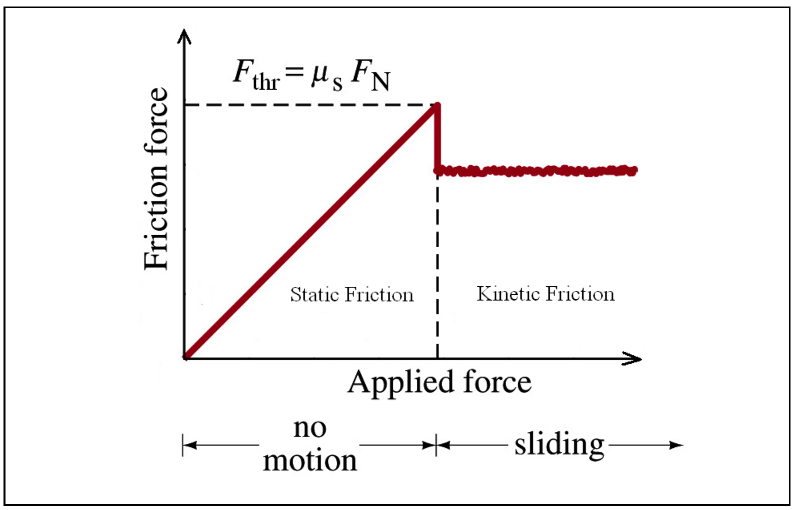
Properties of a friction force (labeled $\mathbf{F}_{\rm fr}$) are

- the friction force $\mathbf{F}_{\rm fr}$ always opposes motion,
- when an object is **moving**, the magnitude of the friction force $F_{\rm fr}$ is proportional to the magnitude of the normal force F_N (even though $\mathbf{F}_{\rm fr} \perp \mathbf{F}_N$), with the constant of proportionality called the coefficient of **kinetic** friction μ_k,
- when the object is **stationary**, the maximum friction force $F_{\rm thr}$ is proportional to the normal force F_N, with the constant of proportionality called the coefficient of **static** friction μ_s,
- in general, the coefficient of static friction μ_s is greater than the coefficient of kinetic friction μ_k: $\mu_s \geq \mu_k$.

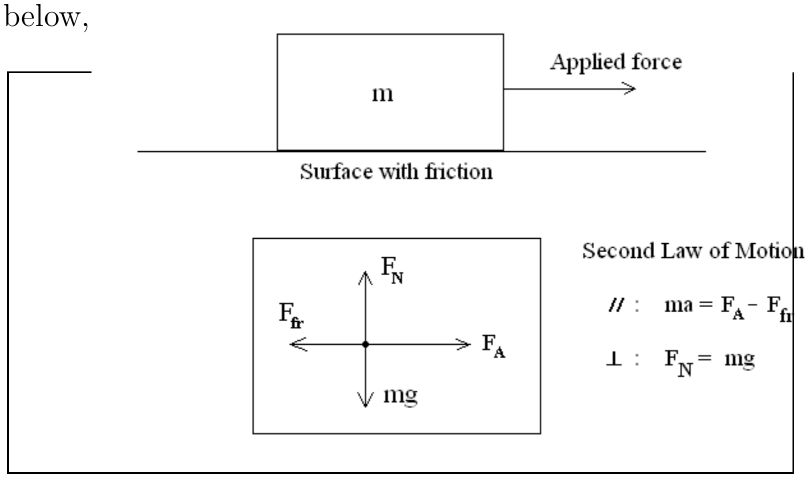
As a simple example, we consider the motion of a box of mass m being dragged on a horizontal surface with coefficients of kinetic friction μ_k and static friction μ_s by an applied force F_A which increases progressively from zero.

At first, the applied force F_A is zero, the box is at rest and, consequently, the force of friction is absent. As the applied force F_A begins to increase ($F_A \neq 0$) but remains below the threshold value $F_{\rm thr} = \mu_s F_N$, where the normal force in this case is equal to the weight of the box $F_N = mg$, the object remains at rest although the force of static friction increases with the applied force.

As the applied force crosses the threshold value $F_A > F_{\rm thr}$, however, the box experiences a *jerky* start and begins to move (see Figure below; Figure 5-3 from Giancoli).



After motion has been initiated, the box is accelerated and, from the free-body force diagram shown below,

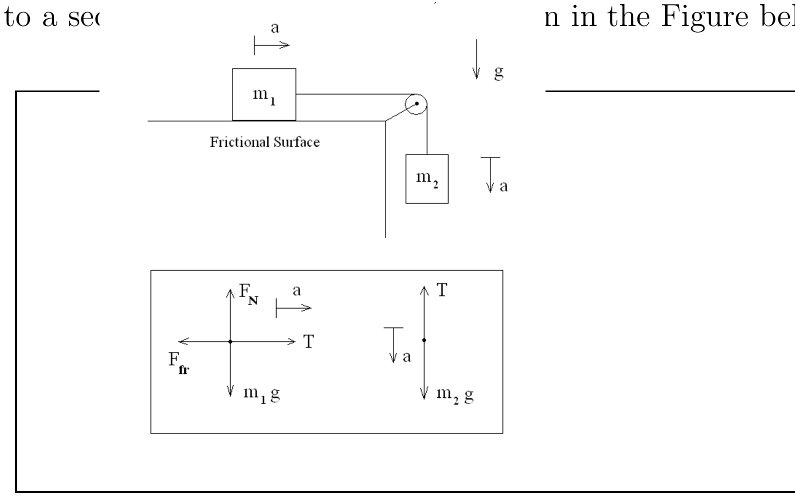


we find that the force of friction is now $F_{\rm fr} = \mu_k \, F_N$ and the system acceleration is

$$m\,a \;=\; F_A \;-\; \mu_k\, mg \quad\rightarrow\quad a \;=\; \frac{F_A}{m} \;-\; \mu_k\, g.$$

As expected, the system acceleration a is less than the acceleration F_A/m produced by the applied force in the absence of friction. Note that after the motion has been initiated, the force of kinetic friction is constant (for low velocities) and is less than the threshold value $\mu_s\, mg$.

At this point, we may redo the force problems that were solved in the absence of friction (e.g., motion on an inclined plane); see examples 5-5 to 5-8. For example, we consider the problem in which an object of mass m_1 sliding with kinetic friction on a horizontal surface while attached to a se n in the Figure below).



The equations representing Newton's Second Law for this system are

$$
\begin{aligned}
0 &= F_N \;-\; m_1\, g \\
m_1\, a &= T \;-\; \mu_k\, F_N \;=\; T \;-\; \mu_k\, m_1 g \\
m_2\, a &= m_2\, g \;-\; T,
\end{aligned}
$$

where a is the system acceleration and T is the tension in the massless string. Solving for the system acceleration, we find

$$
a \;=\; \left(\frac{m_2 - \mu_k\, m_1}{m_2 + m_1} \right)\; g,
$$

while the string tension is found to be

$$
T \;=\; m_2\,(g - a) \;=\; \frac{m_2\, g}{m_2 + m_1} \Big((m_2 + m_1) \;-\; (m_2 - \mu_k\, m_1) \Big) \;=\; \left(\frac{m_2\, m_1}{m_2 + m_1} \right)\; g\,(1 + \mu_k).
$$

Note that, as expected, the system acceleration is reduced by friction while the string tension is increased.

- **Forces due to Uniform Circular Motion**

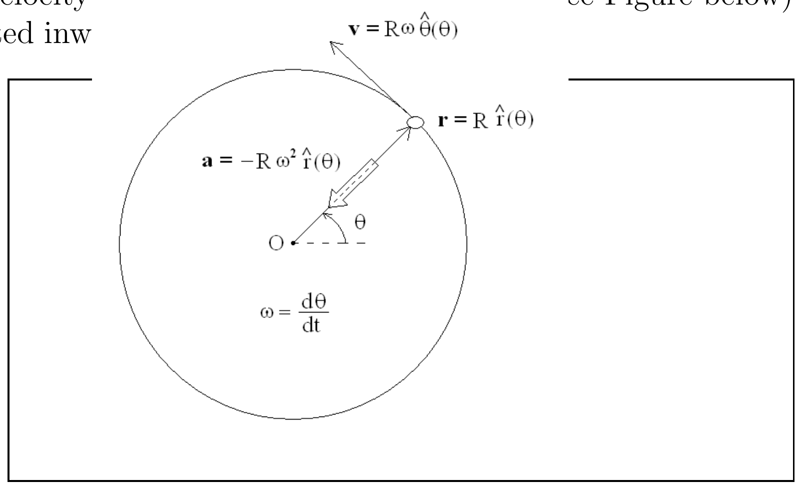
When an object of mass m undergoes uniform circular motion at a constant distance R from a center of rotation O, the position of the object can be expressed as

$$\mathbf{r}(\theta) \;=\; R\;(\cos\theta\,\widehat{\mathsf{x}} \;+\; \sin\theta\,\widehat{\mathsf{y}}) \;=\; R\,\widehat{r}(\theta),$$

where the angle θ is measured from the x-axis and $\omega = d\theta/dt$ denotes the constant angular speed. From this expression, we obtain the following expressions for the velocity $\mathbf{v} = d\mathbf{r}/dt$ and the acceleration $\mathbf{a} = d\mathbf{v}/dt$:

$$\begin{aligned} \mathbf{v} \;&=\; \omega\,\frac{d\mathbf{r}}{d\theta} \;=\; R\,\omega\;(-\,\sin\theta\,\widehat{\mathsf{x}} \;+\; \cos\theta\,\widehat{\mathsf{y}}) \;=\; R\,\omega\;\widehat{\theta}(\theta), \\ \mathbf{a} \;&=\; \omega\,\frac{d\mathbf{v}}{d\theta} \;=\; -\,R\,\omega^2\;(\cos\theta\,\widehat{\mathsf{x}} \;+\; \sin\theta\,\widehat{\mathsf{y}}) \;=\; -\,R\,\omega^2\;\widehat{r}(\theta). \end{aligned}$$

Note that the velocity vector is tangential to the circle (see Figure below) while the acceleration is directed inw



This inward acceleration is known as the *centripetal* acceleration $\mathbf{a}_c$, with constant magnitude

$$a_c \;=\; R\,\omega^2 \;=\; v\,\omega \;=\; \frac{v^2}{R}.$$

Note that, according to Newton's First and Second Laws, an object undergoing uniform circular motion must be under the influence of an inward centripetal force $\mathbf{F}_c = m\,\mathbf{a}_c$.

This centripetal force, however, must have a physical origin through either the force of gravity (e.g., the uniform circular motion of the Moon around Earth), the force of friction (e.g., when a car takes a sharp turn on the road), the normal force provided by a surface (e.g., a roller-coaster ride going through a loop), or the force of tension (e.g., when a rock tied to a string is whirled along a horizontal or vertical circle).

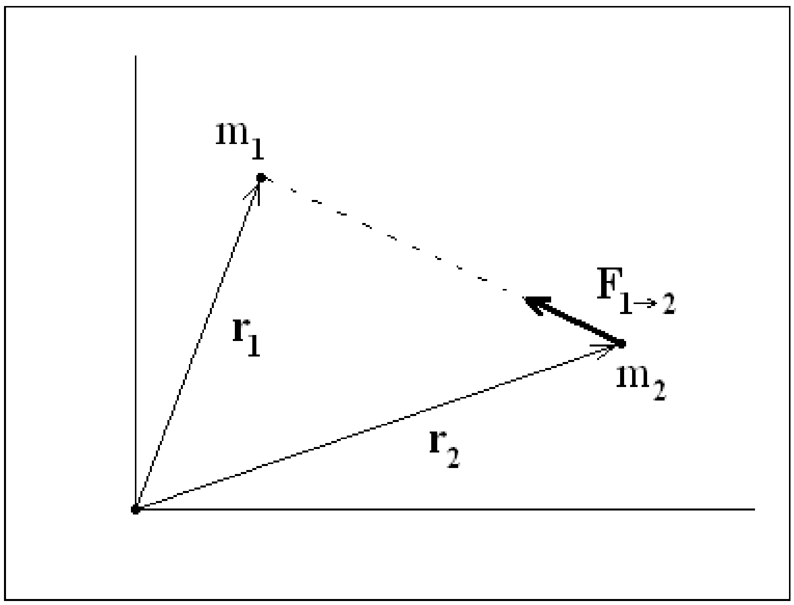
Test your knowledge: Problems 2, 4-5, 35 & 37 of Chapter 5

7 Newton's Law of Universal Gravitation

Textbook Reference: Chapter 6 – sections 1-6.

- **Law of Universal Gravitation**

Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them. This force acts along the line joining the two particles.



$$\mathbf{F}_{1\to 2} \;=\; -\,G\,m_1 m_2\,\frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

The gravitational attraction g_E of an object of mass m at rest on the Earth's surface

$$g_E \;=\; G\,\frac{m_E}{R_E^2} = 9.80 \text{ m} \cdot \text{s}^{-2}$$

is, therefore, inversely proportional to the square of Earth's radius $R_E = 6.38 \times 10^6$ m and proportional to Earth's mass $m_E = 5.98 \times 10^{24}$ kg. Note that, since Jupiter is 317.8 times more massive than Earth and it's radius is 11.2 times larger than Earth's radius, the gravitational acceleration at Jupiter's surface is $g_J = 2.5\ g_E = 24.8$ m$\cdot$s^{-2}.

- **Orbiting Satellites**

A satellite of mass m_S orbiting the Earth at a distance R from its center will complete its orbit (assumed to be circular) after a period T; its orbital speed is, therefore, $v = 2\pi R/T$, which is assumed constant. In order to perform its uniform circular motion, the centripetal acceleration v^2/R of the satellite must come from the Earth's gravitational acceleration $G\,m_E/R^2$ and, thus, we find the ratio

$$\frac{R^3}{T^2} \;=\; \frac{G\,m_E}{4\pi^2} \;=\; 1.01 \;\times\; 10^{13}\ \mathrm{m}^3\cdot\mathrm{s}^{-2}.$$

This law implies that the orbital period T of a satellite increases with the orbital radius R as $R^{3/2}$. A *geosynchronous* (GS) orbit is in one for which the period is one (Earth) day $T_{GS} = 8.64 \times 10^4$ s, from which we get the geosynchronous radius $R_{GS} = 4.22 \times 10^7$ m $=$ $6.6\ R_E$.

• Kepler's Laws of Planetary Motion

Kepler's First Law: Each planet moves an elliptical path about the Sun (which occupies one of its foci).

This law is explained by Newton's inverse-square relation between force and distance.

Kepler's Second Law: Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal periods of time.

This law is explained by Newton's conservation law of *angular momentum* $v_1\,r_1 = v_2\,r_2$.

Kepler's Third Law: The ratio R^3/T^2 is a constant for all planets orbiting the Sun.

Newton's universal theory predicts that

$$\frac{R^3}{T^2} \;=\; \frac{G\,m_\odot}{4\pi^2} \;=\; 3.38 \;\times\; 10^{18}\ \mathrm{m}^3\cdot\mathrm{s}^{-2}.$$

If we introduce the *astronomical unit* (AU) defined as the average Earth-Sun distance, 1 AU $= 1.50 \times 10^{11}$ m, as a unit of distance and the Earth's orbital period, 1 y $= 3.16 \times 10^7$ s, as unit of time then Kepler's Third Law becomes $R(\mathrm{AU})^3 = T(\mathrm{y})^2$.

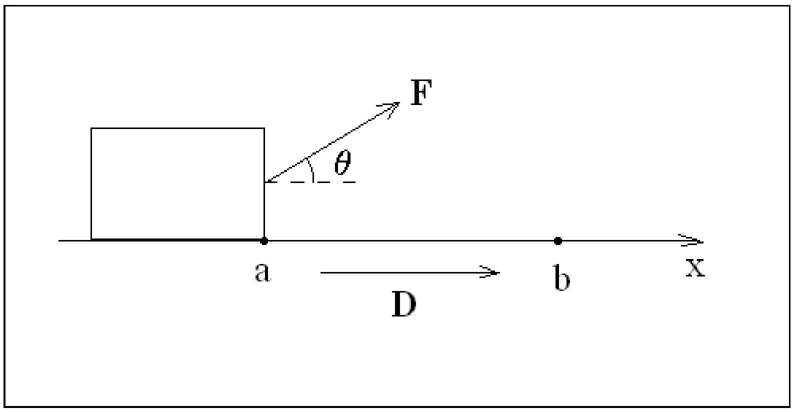
Test your knowledge: Problems 1, 3, 18 & 35 of Chapter 6

8 Work-Energy Theorem I

Textbook Reference: Chapter 7 – sections 1-6.

- **Work done by a Constant Force**

The work W done by a constant force $\mathbf{F}$ applied to an object at an angle θ from the horizontal plane while the object is displaced by an amount $\mathbf{D} = D\,\widehat{\mathsf{x}}$ (see Figure below)



is evaluated as

$$W \;=\; \mathbf{F}\cdot\mathbf{D} \;=\; F_x\,D \;=\; F\,D\,\cos\theta,$$

where the scalar (or *dot*) product of two vectors $\mathbf{A} = (A_x, A_y)$ and $\mathbf{B} = (B_x, B_y)$ is defined as

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= A_x\,B_x \;+\; A_y\,B_y \;=\; |\mathbf{A}|\,|\mathbf{B}|\;(\cos\theta_A\,\cos\theta_B \;+\; \sin\theta_A\,\sin\theta_B) \\ &= |\mathbf{A}|\,|\mathbf{B}|\;\cos(\theta_A - \theta_B). \end{aligned}$$

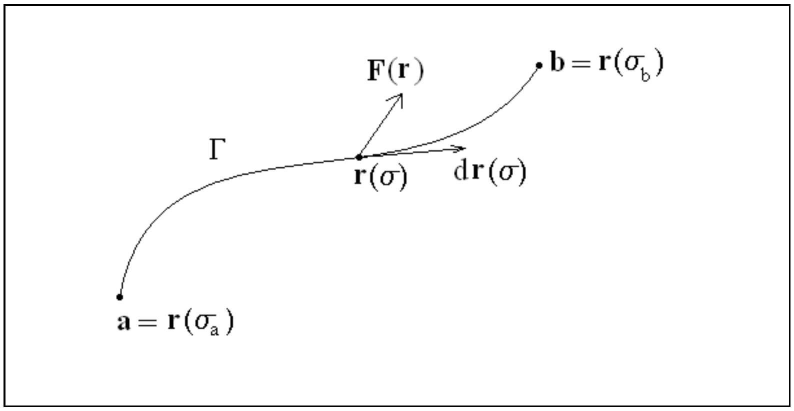
Note from this definition, the work is positive ($\cos\theta > 0$), negative ($\cos\theta < 0$), or zero ($\cos\theta = 0$). For example, since a friction force always opposes motion, the work due to a friction is **always** negative.

The unit of work is the Joule (abbreviated J) and is defined as 1 J = 1 N · m. The net work **done on** an object is defined as the sum of the work done by all forces applied on the object:

$$W_{net} \;=\; \sum_i W_i.$$

- **Work done by a Varying Force**

We now consider the work W_Γ done by a varying force $\mathbf{F}(\mathbf{r})$ (whose magnitude and direction depends on position $\mathbf{r}$) in going from an initial position $\mathbf{a}$ to a final position $\mathbf{b}$ by following a *path* Γ (see Figure below).

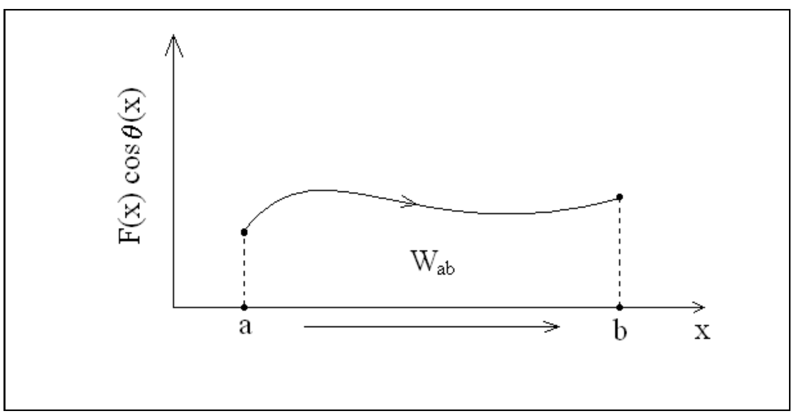


If we introduce a path parameterization $\sigma \rightarrow \mathbf{r}(\sigma)$ for Γ, with $\mathbf{a} = \mathbf{r}(\sigma_a)$ and $\mathbf{b} = \mathbf{r}(\sigma_b)$, then the work W_Γ done by the force is evaluated as a *path* integral

$$W_\Gamma \;=\; \int_{\sigma_a}^{\sigma_b} \mathbf{F}(\mathbf{r}(\sigma)) \cdot \frac{d\mathbf{r}(\sigma)}{d\sigma}\, d\sigma.$$

As a special case, we consider the work done by the force $\mathbf{F} = F_x(x)\,\widehat{\mathsf{x}} + F_y(x)\,\widehat{\mathsf{y}}$ in going from the initial position $(a,0)$ to the final position $(b,0)$ by following a path along the x-axis:

$$W_{ab} \;=\; \int_a^b |\mathbf{F}(x)|\;\cos\theta(x)\; dx$$



i.e., the work W_{ab} done by the varying force is the area under the curve $|\mathbf{F}(x)|\;\cos\theta(x)$ from $x = a$ to $x = b$.

- **Work done on a Spring**

The applied force $\mathbf{F}_{app}$ necessary to keep a spring elongated or stretched by an amount x is directly proportional to x: $\mathbf{F}_{app} = k\,\mathbf{x}$, where k denotes the spring constant (representing

the stiffness of the spring) and $\mathbf{x}$ denotes the spring displacement. The spring itself exerts a force $\mathbf{F}_{sp} = -\,\mathbf{F}_{app}$ in the opposite direction (Hooke's Law):

$$\mathbf{F}_{sp}(\mathbf{x}) \;=\; -\,k\,\mathbf{x},$$

i.e., the spring opposes its compression or its elongation. The work done by an applied force $\mathbf{F} = -\,kx\,\widehat{\mathsf{x}}$ to compress a spring ($d\mathbf{r} = -\,dx\,\widehat{\mathsf{x}}$) by an amount X is, therefore, given as

$$W(X) \;=\; \int_0^X \;(-\,kx\,\widehat{\mathsf{x}})\cdot(-\,dx\,\widehat{\mathsf{x}}) \;=\; \int_0^X \;kx\;dx \;=\; \frac{k}{2}\,X^2.$$

You can easily convince yourself that the work done by an applied force to elongate a spring by an amount X is also $W = \frac{1}{2}\,kX^2$.

- **Kinetic Energy and Net Work done on an Object**

The net work W_{net} done on an object of mass m causes it to accelerate or deccelerate from an initial velocity $\mathbf{v}_i$ to a final velocity $\mathbf{v}_f$. If we introduce the *kinetic* energy, denoted K, associated with motion and defined for an object of mass m and velocity $\mathbf{v}$ as

$$K \;=\; \frac{m}{2}\,|\mathbf{v}|^2,$$

then the **First Work-Energy Theorem** states that

The net work done on an object is equal to the change in its kinetic energy.

$$W_{net} \;=\; \Delta\,K \;=\; K_f \;-\; K_i \;=\; \frac{m}{2}\,\left(|\mathbf{v}_f|^2 \;-\; |\mathbf{v}_i|^2\right).$$

Hence, the object accelerates if $W_{net} > 0$, or deccelerates if $W_{net} < 0$, or its kinetic energy is constant if $W_{net} = 0$.

Test your knowledge: Problems 1, 3, 5, & 43 of Chapter 7

9 Work-Energy Theorem II

Textbook Reference: Chapter 8 – sections 1-8.

- **Work done by Conservative Forces**

The work W_{ab} done by a **conservative** force is **independent** of the path taken from an initial point **a** to a final point **b** and depends only on conditions at the initial and final points. These conditions are described in terms of the **potential** energy U, which is the energy associated with the position or configuration of an object as it interacts with its *environment*.

The work W_{ab} done by a conservative force $\mathbf{F}$ is, therefore, defined in terms of the potential energy difference $\Delta U = U_b - U_a$ as

$$W_{ab} \;=\; \int_a^b \mathbf{F} \cdot d\mathbf{r} \;=\; -\,\Delta U.$$

This definition implies that work must be done against a conservative force in order to increase the potential energy of an object. Note that since only the difference in potential energy is associated with work, the choice of where $U = 0$ is **arbitrary** and can be chosen wherever it is most convenient. In one-dimensional problems with conservative forces $F(x)$, for example, the potential energy $U(x)$ can be written

$$U(x) \;=\; -\int_a^x F(s)\,ds,$$

where the point a is chosen so that $U(a) = 0$.

There are several types of conservative forces and their associated potential energies. In this Chapter, we investigate only two types of conservative forces: (a) the gravitational forces near Earth and due to a massive object and (b) the elastic force due to a spring. First, assuming that a gravitation *center* of mass M is located at the origin ($r = 0$), the gravitational force on an object of mass m located at a distance r from the gravitation center is

$$\mathbf{F}_G \;=\; -\,\frac{G\,mM}{r^2}\,\widehat{\mathsf{r}} \;=\; F(r)\,\widehat{\mathsf{r}},$$

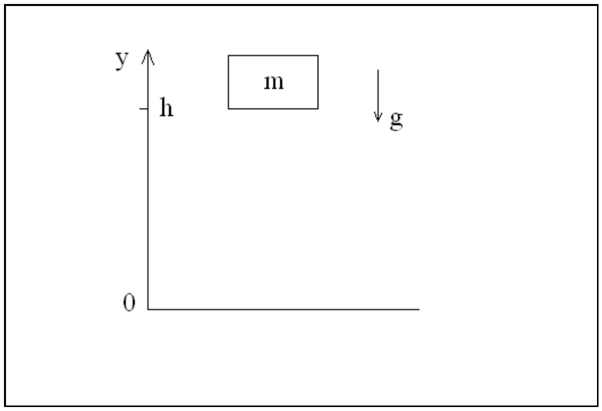
so that the work done by the gravitational force in bringing the object from $r = \infty$ to a distance $r = R$ from the gravitation center is

$$W_{\infty R} \;=\; \int_\infty^R \mathbf{F}_G \cdot (\widehat{\mathsf{r}}\,dr) \;=\; -\,G\,mM \int_\infty^R \frac{dr}{r^2} \;=\; \frac{G\,mM}{R}.$$

Since $W_{\infty R} \;=\; -\,U_G(R) \;+\; U_G(\infty)$, by definition of the potential energy, and taking $U_G(\infty) = 0$, we, therefore, find an expression for the gravitational potential energy $U_G(r)$ for an object of mass m located at a distance

$$U_G(r) \;=\; -\,\frac{G\,mM}{r} \;=\; -\int_\infty^r \left(-\,\frac{G\,mM}{s^2}\right) ds.$$

We may now apply this formula for gravitational potential energy to the case of the gravitational potential energy near Earth. When an object of mass m is placed at a height h above the ground and let go (see Figure below), Earth's gravitational force causes the object to move from the initial position $R_E + h$ to the final position R_E, where $R_E = 6.38 \times 10^6$ m denotes the radius of the Earth.



The work done by gravity is, therefore, calculated as

$$W \;=\; -\,U_G(R_E) \;+\; U_G(R_E+h) \;=\; G\,mM_E\left(\frac{1}{R_E} \;-\; \frac{1}{R_E+h}\right) \;=\; \frac{G\,mM_E\,h}{R_E\,(R_E+h)}.$$

If $h \ll R_E$ (e.g., near-Earth conditions), then $R_E + h \simeq R_E$ in the denominator above and using the definition $g = GM_E/R_E^2$ for the near-Earth gravitational acceleration, we find an expression for the gravitational potential energy $U_g(h)$ of the object at height h above ground given as

$$U_g(h) \;=\; mg\,h \;=\; -\int_0^h (-\,mg)\;ds,$$

where we now assume that the gravitational potential energy vanishes at ground level. Note that, with this definition, any position **below** ground level would have **negative** potential energy.

As a second type of conservative force, we consider the elastic force associated with the elongation or compression of a spring (with constant k). Since a spring always opposes changes in its equilibrium length with a force $F_S(x) = -\,kx$, where x denotes the displacement from equilibrium ($x > 0$ for elongation and $x < 0$ for compression), then the elastic potential energy $U_S(X)$ associated with a displacement X is

$$U_S(X) \;=\; -\int_0^X (-\,ks)\;ds \;=\; \frac{k}{2}\,X^2,$$

which implicitly assumes that the elastic potential energy associated with a spring of constant k is zero at equilibrium.

- **Work-Energy Theorem in the presence of Conservative Forces**

When an object of mass m is only exposed to conservative forces, the net work done on the object by these forces is

$$W_{net} \;=\; W_{\rm C} \;=\; -\,\Delta U,$$

where U is the sum of all potential energies. From the First Work-Energy Theorem, which states that the net work done on the object is equal to the change in its kinetic energy, $W_{net} = \Delta K$, we obtain the **Energy Conservation Law**:

$$\Delta K \,+\, \Delta U \,=\, 0,$$

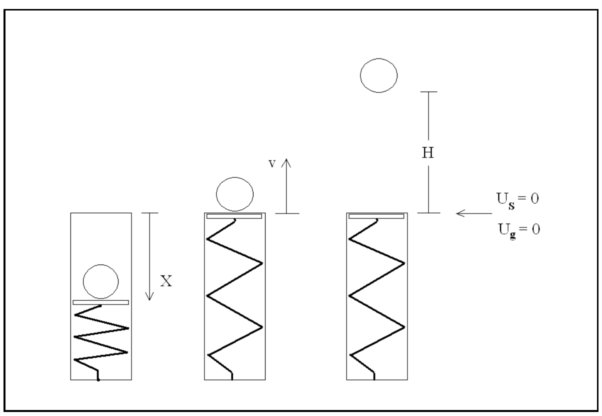
which means that the total mechanical energy $E = K + U$ of the object is conserved.

Hence, for example, when an object of mass m is initially at rest at a height h above ground, its initial kinetic energy is $K_i = 0$, its initial potential energy is $U_i = mgh$, and its initial mechanical energy is $E_i = 0 + mgh = mgh$. After the object is released, it **converts** potential energy into kinetic energy and, just before it hits the ground, its final kinetic energy is $K_f = \frac{1}{2}\,mv_f^2$, where v_f denotes its final speed, its final potential energy is U_f and its final mechanical energy is $E_f = \frac{1}{2}\,mv_f^2 + 0 = \frac{1}{2}\,mv_f^2$. By conservation of mechanical energy, $E_f = E_i$, we find

$$\frac{1}{2}\,mv_f^2 \;=\; mg\,h \quad\rightarrow\quad v_f \;=\; \sqrt{2\,gh},$$

which is what we would obtain by solving the free-fall equations of motion: $y = h - \frac{1}{2}\,gt^2$ and $v_y = -\,gt$.

As a second example, we consider the problem of finding the maximum height H achieved by an object of mass m placed on a spring of constant k compressed by a distance X (see Figure below).



Here, we take the gravitational and elastic potential energies to be both zero at the mouth of the barrel (see Figure above). Moreover, we note that the initial and final kinetic energies are both zero. The initial potential energy is the sum of the initial gravitational energy $U_{gi} = -\,mgX$ (i.e., the initial position is below the zero-potential line) and the initial elastic potential energy $U_{Si} = \frac{1}{2}\,kX^2$, so that

$$U_i \;=\; U_{gi} \;+\; U_{Si} \;=\; -\,mg\,X \;+\; \frac{k}{2}\,X^2.$$

The final potential energy, however, is only gravitational with

$$U_f \;=\; U_{gf} \;=\; mg\,H,$$

so that conservation of energy yields

$$H \;=\; \frac{k\,X^2}{2\,mg} \;-\; X.$$

Note that at the mouth of the barrel, the object is moving with upward velocity

$$v \;=\; \sqrt{2\,gH} \;=\; \sqrt{\frac{k\,X^2}{m} \;-\; 2\,gX}.$$

Lastly, an object of mass m and moving with velocity $\mathbf{v}$ may escape the attraction of a gravitational center of mass M if its mechanical energy is positive

$$E \;=\; \frac{m}{2}\,|\mathbf{v}|^2 \;-\; \frac{G\,mM}{r} \;>\; 0.$$

By conservation of energy, an object at the surface of a massive object of mass M and radius R will be able to escape (at infinity) if its radial velocity exceeds the escape velocity

$$v_{\rm esc} \;=\; \sqrt{\frac{2\,GM}{R}}.$$

For example, the escape velocity for a rocket leaving the surface of Earth is 11.2 km/s while it is 2.4 km/s if the rocket were to leave the surface of the Moon (i.e., 4.7 times smaller). Hence, the kinetic energy needed for the rocket to leave Earth will be about $(4.7)^2 \simeq 22$ times larger than the kinetic energy needed to leave the surface of the Moon.

- **Work-Energy Theorem in the presence of Non-Conservative Forces**

Non-conservative forces are defined as forces for which their associated work integrals are path-dependent. Two types of non-conservative forces are considered in this course: applied forces and friction forces. In both cases, the work path integal is proportional to the path length; in particular, the work due to friction as an object of mass m is moved on a rough horizontal surface from point $\mathbf{a}$ to point $\mathbf{b}$ is

$$W_{\rm fr} \;=\; -\,\mu_k\,mg\,\int_{\mathbf{a}}^{\mathbf{b}}\,ds,$$

which is always negative since friction opposes motion. By defining the work done by non-conservative forces as $W_{\rm NC} = W_{\rm app} + W_{\rm fr}$, the net work done on an object by conservative and non-conservative forces is

$$W_{net} \;=\; W_{\rm C} \;+\; W_{\rm NC} \;=\; -\,\Delta U \;+\; W_{\rm app} \;+\; W_{\rm fr},$$

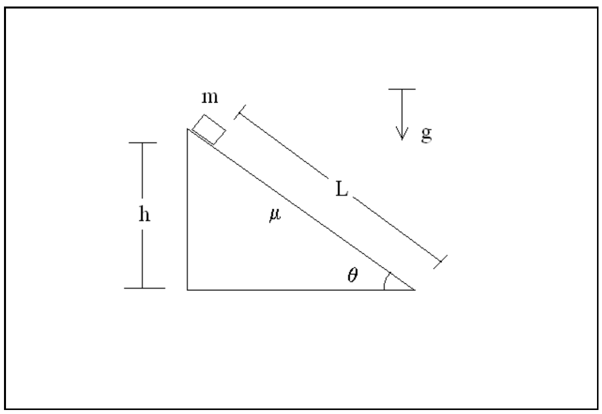
so that the First Work-Energy Theorem, $W_{net} = \Delta K$, now yields the **Second Work-Energy Theorem**

$$\Delta K \;+\; \Delta U \;=\; \Delta E \;=\; W_{\rm NC} \;=\; W_{\rm app} \;+\; W_{\rm fr},$$

which states that

The change in the total mechanical energy of an object is due to work associated with non-conservative foces.

As an example of the Second Work-Energy Theorem, we consider the problem of an object of mass m sliding down a rough inclined plane (making an angle θ with respect to the horizontal plane) over a distance L (see Figure below).



In sliding down the inclined plane, the object decreases its gravitational potential energy, $\Delta U = -\,mg{\rm L}\sin\theta$, where $h = L\,\sin\theta$ denotes the drop in height. Next, since the initial kinetic energy is zero and its final kinetic energy is $K_f = \frac{1}{2}\,mv_f^2$, we find $\Delta K = \frac{1}{2}\,mv_f^2$, while the work due to friction is $W_{\rm fr} \;=\; -\;F_{\rm fr}\,L \;=\; -\;\mu_k\,mgL\,\cos\theta$. Hence, the Second Work-Energy Theorem implies that

$$\frac{1}{2}\,mv_f^2 \;-\; mg\,L\sin\theta \;=\; -\,\mu_k\,mgL\,\cos\theta,$$

and, for example, the final velocity is

$$v_f \;=\; \sqrt{2\,gL\,(\sin\theta \;-\; \mu_k\,\cos\theta)},$$

which is less than $\sqrt{2\,gh}$ because of frictional effects.

Another statement of the energy conservation law is based on the definition $Q = -\,W_{\rm fr}$ for the thermal energy produced as a result of friction so that, in the absence of applied forces ($W_{\rm app} = 0$), the **generalized** energy conservation law becomes

$$\Delta K \;+\; \Delta U \;+\; Q \;=\; 0.$$

Hence, when an object is initially moving on a rough horizontal surface (with $\Delta U = 0$) and comes to rest as a result of friction, the mechanical energy is dissipated in the form of heat $Q = -\,\Delta K > 0$, with both object and surface exhibiting increases in thermal energy (see Chapter 19 for further details).

Lastly, we note that, in the absence of conservative forces, the net work done on an object of mass m is proportional to the net acceleration **a** and the total displacement **D** covered by the object, $W_{net} = m\,\mathbf{a}\cdot\mathbf{D}$, and the Second Work-Energy Theorem yields

$$v_f^2 \;=\; v_i^2 \;+\; 2\,\mathbf{a}\cdot\mathbf{D}.$$

- **Power**

The instantaneous power associated with a process able to do work is defined as the rate at which work is performed, i.e.,

$$P \;=\; \frac{dW}{dt},$$

which also defines the rate at which energy is **transformed**. The unit of power is the Watt (W), defined as

$$1\ \mathrm{W} \;=\; 1\ \mathrm{J}\cdot\mathrm{s}^{-1}.$$

A horsepower (hp) is defined as 1 hp = 746 W. For example, when a 1,000 kg autmobile accelerates from 0 m/s to 30 m/s (108 km/h) in 3 seconds, the change in kinetic energy is ΔK = 450 kJ and the engine must develop a minimum average power of 150 kW (or approximately 200 hp); the inclusion of air resistance and other dissipative effects raises this value.

Test your knowledge: Problems 3, 11, 54 & 57 of Chapter 8

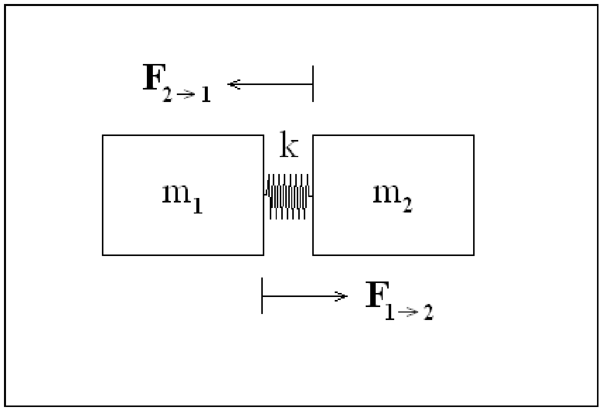
10 Linear Momentum and Collisions

Textbook Reference: Chapter 9 – sections 1-9.

- **Linear Momentum and Newton's Second and Third Laws**

Consider two objects of masses m_1 and m_2 attached together with a spring of constant k. Through the compressed spring, the object of mass m_1 exerts a force $\mathbf{F}_{1\to2}$ on the object

of mass m_2 while, in return, the object of mass m_2 exerts a force $\mathbf{F}_{2\to 1}$ on the object of mass m_1.



The two objects are initially at rest. When the spring is released, the two objects move apart from each other with opposing velocities $\mathbf{v}_1$ (to the right) and $\mathbf{v}_2$ (to the left). These *final* velocities are obtained as a result of the respective (averaged) forces experienced by the objects

$$\mathbf{F}_{1\to 2} \;=\; m_2\,\frac{\Delta\mathbf{v}_2}{\Delta t} \;=\; -\,m_1\,\frac{\Delta\mathbf{v}_1}{\Delta t} \;=\; -\,\mathbf{F}_{2\to 1},$$

where we have made use of Newton's Second and Third Laws of motion. Since the time intervals Δt over which the two objects *interact* with each other are identical, Newton's Third Law implies that

$$\Delta\,(m_1\,\mathbf{v}_1 \;+\; m_2\,\mathbf{v}_2) \;=\; 0,$$

which means that the quantity

$$\mathbf{P} \;=\; \sum_i \mathbf{p}_i \;=\; \sum_i m_i\,\mathbf{v}_i$$

is conserved throughout the process. The vector quantity $\mathbf{P}$ is called the total **linear momentum** (or total momentum), and Newton's Second Law is now expressed as

$$\mathbf{F}_{net} \;=\; \sum_i \mathbf{F}_i \;=\; \sum_i \frac{d\mathbf{p}_i}{dt} \;=\; \frac{d\mathbf{P}}{dt}.$$

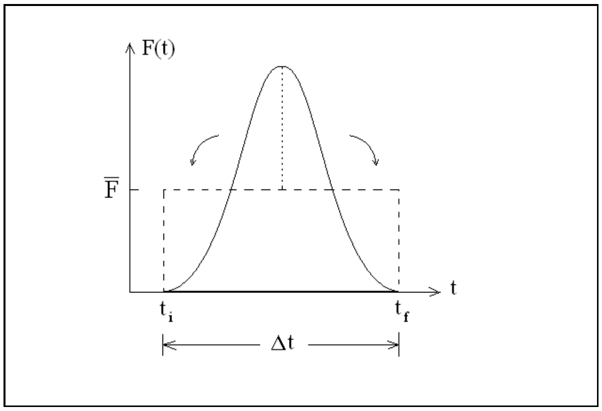
Hence, if the (external) net force acting on a system of particles is zero, then the total momentum of the system is conserved; this is the **Law of Conservation of Momentum**.

- **Collisions and Impulse**

When two objects, moving with respective momentum $\mathbf{p}_1$ and $\mathbf{p}_2$, collide, the objects transfer momentum to each other. The change in momentum $\Delta\mathbf{p}_1$ for particle 1 is obtained integrating the force $\mathbf{F}_{2\to 1}$ through the collision interval Δt:

$$\Delta\mathbf{p}_1 \;=\; \int_{t_i}^{t_f} \mathbf{F}_{2\to 1}(t)\,dt \;=\; \mathbf{J}_{2\to 1},$$

where the collision begins at time t_i and ends at time $t_f = t_i + \Delta t$ and $\mathbf{J}_{2\rightarrow 1}$ denotes the **impulse** received from particle 2. The impulse is clearly defined as the area under the force-versus-time curve (see Figure below).



From this definition, we may easily extract an average force of **impact** $\overline{\mathbf{F}}$ defined as

$$\overline{\mathbf{F}} \;=\; \frac{\Delta\mathbf{p}}{\Delta t}.$$

For example, when a 1 kg ball hits a vertical wall with velocity $\mathbf{v}_i = (10\,\mathrm{m/s})\,\widehat{\mathsf{x}}$ and comes to rest on the wall in a time of 0.01 sec, the wall has exerted an average force of impact of

$$\overline{\mathbf{F}} \;=\; \frac{1\,\mathrm{kg}\;(0\,\mathrm{m/s} - 10\,\mathrm{m/s})}{0.01\,\mathrm{s}}\,\widehat{\mathsf{x}} \;=\; -\;(1,000\,\mathrm{N})\,\widehat{\mathsf{x}}.$$

If the ball bounces back with velocity $\mathbf{v}_f \;=\; -\,(10\,\mathrm{m/s})\,\widehat{\mathsf{x}}$, however, the average force of impact is now $\overline{\mathbf{F}} = -\;(2,000\,\mathrm{N})\;\widehat{\mathsf{x}}$. Note that the ball exerts a force of impact **on** the wall of equal magnitude but in the opposite direction.

For a given change in momentum $\Delta\mathbf{p}$, we find that the relation

$$\Delta\mathbf{p} \;=\; \overline{\mathbf{F}}\,\Delta t$$

implies that a **short** collision time is associated with a **large** force of impact while a **long** collision time is associated with a **small** force of impact.

- **Elastic and Inelastic Collisions**

All collisions conserve the total momentum of the colliding particles. For collisions involving two particles (labeled 1 and 2), we then find that the total momentum before the collision $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and the total momentum after the collision $\mathbf{P}' = \mathbf{p}_1' + \mathbf{p}_2'$ (a prime is used to denote quantities after the collision) are equal

$$\mathbf{p}_1 \;+\; \mathbf{p}_2 \;=\; \mathbf{p}_1' \;+\; \mathbf{p}_2' \quad\rightarrow\quad \Delta\mathbf{p}_1 \;=\; -\;\Delta\mathbf{p}_2.$$

An *elastic* collision is one in which the total kinetic energy of the colliding particles is also conserved:

$$K \;=\; K_1 \;+\; K_2 \;=\; K_1' \;+\; K_2' \;=\; K'.$$

An *inelastic* collision is one in which only momentum is conserved. All collisions in which particles stick to each other after the collision are inelastic. For example, when a particle of mass m_1 moving at speed v collides with a second particle of mass m_2 (initially at rest) and the two particles remain together after the collision, the conservation of momentum dictates that

$$m_1\, v \;=\; (m_1 + m_2)\, v',$$

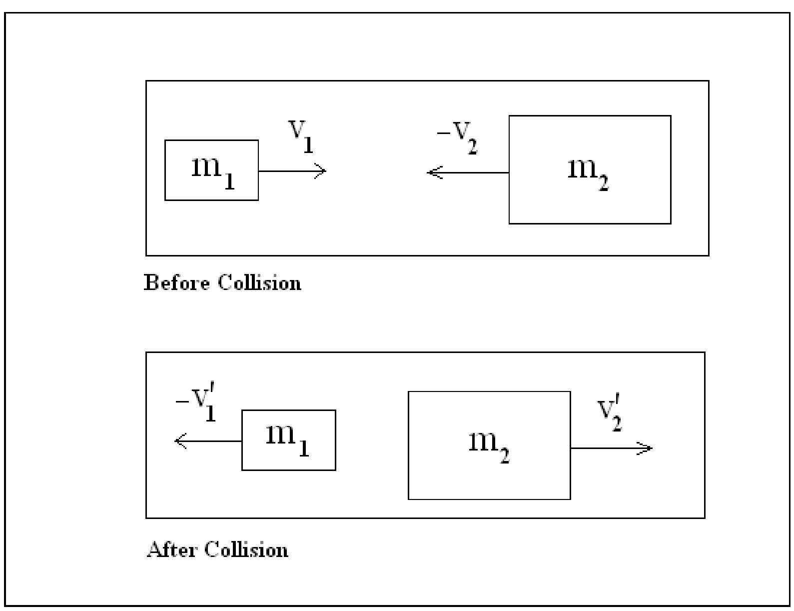
where v' is the speed of the two-particle system of combined mass $m_1 + m_2$. The kinetic energy before the collision is $K = \frac{1}{2}\, m_1 v^2$ while the kinetic energy after the collision is

$$K' \;=\; \frac{1}{2}\,(m_1 + m_2)\, v'^2 \;=\; \frac{1}{2}\,(m_1 + m_2) \left(\frac{m_1\, v}{m_1 + m_2} \right)^2 \;=\; \left(\frac{m_1}{m_1 + m_2} \right)\; K.$$

Hence, since $K' < K$, kinetic energy was lost in the course of the *sticking* collision; most likely the energy was converted into thermal energy, sound energy, and other forms of energy.

• Elastic Collisions in One Dimension

To investigate elastic collisions in one dimension, we consider the generic scenario in which the conditions before and after the collision are as follows; here, we shall assume that an object moving to the right is moving with positive velocity. Before the collision, object 1 (mass m_1) is moving to the right with velocity v_1 while object 2 (mass m_2) is moving to the left with velocity $-\,v_2$. After the collision, object 1 is moving to the left with velocity $-\,v_1'$ while object 2 is moving to the right with velocity v_2' (see Figure below).



Conservation of momentum implies that

$$m_1 v_1 \;-\; m_2 v_2 \;=\; -\, m_1 v_1' \;+\; m_2 v_2',$$

or, introducing the mass ratio $\alpha = m_1/m_2$ and the velocity changes $\Delta v_1 = -\, v_1' - v_1 < 0$ and $\Delta v_2 = v_2' - (-\, v_2) > 0$, we find

$$\Delta v_2 \;=\; -\,\alpha\, \Delta v_1 \quad \rightarrow \quad \left\{ \begin{array}{rcl} v_1' & = & -\, v_1 \;-\; \Delta v_1 \\ \\ v_2' & = & -\, v_2 \;-\; \alpha\, \Delta v_1 \end{array} \right.$$

Conservation of kinetic energy, on the other hand, implies that

$$\frac{m_1}{2}\, v_1^2 + \frac{m_2}{2}\, v_2^2 \;=\; \frac{m_1}{2}\, v_1'^{\,2} + \frac{m_2}{2}\, v_2'^{\,2} \quad \rightarrow \quad \alpha\, v_1^2 \,+\, v_2^2 \;=\; \alpha\,(v_1 + \Delta v_1)^2 \,+\, (v_2 + \alpha\, \Delta v_1)^2 \,.$$

By expanding the right side of the second equation and cancelling terms on the left side, we obtain

$$0 \;=\; \alpha\, \Delta v_1 \,[\, 2\,(v_1 + v_2) \;+\; \Delta v_1\,(1+\alpha)\,]\,,$$

whose solution is either $\Delta v_1 = 0$ (i.e., nothing happened) or

$$\Delta v_1 \;=\; -\, \frac{2\,(v_1 + v_2)}{(1+\alpha)}.$$

From this solution, we obtain the speeds after the collision

$$v_1' \;=\; -\, v_1 \;-\; \Delta v_1 \;=\; \frac{-\, v_1\,(1+\alpha) \;+\; 2\,(v_1 + v_2)}{(1+\alpha)} \;=\; \frac{v_1\,(1-\alpha) \;+\; 2\, v_2}{(1+\alpha)}$$

$$v_2' \;=\; -\, v_2 \;-\; \alpha\, \Delta v_1 \;=\; \frac{-\, v_2\,(1+\alpha) \;+\; 2\alpha\,(v_1 + v_2)}{(1+\alpha)} \;=\; \frac{2\alpha\, v_1 \;+\; v_2\,(\alpha - 1)}{(1+\alpha)}$$

By restoring masses, we find

$$v_1' \;=\; \frac{v_1\,(m_2 - m_1) \;+\; 2 m_2\, v_2}{(m_1 + m_2)} \quad \text{and} \quad v_2' \;=\; \frac{2 m_1\, v_1 \;-\; v_2\,(m_2 - m_1)}{(m_1 + m_2)}.$$

If the object 2 is initially at rest, for example, these equations simplify to

$$v_1' \;=\; \frac{v_1\,(m_2 - m_1)}{(m_1 + m_2)} \quad \text{and} \quad v_2' \;=\; \frac{2 m_1\, v_1}{(m_1 + m_2)},$$

and, hence, object 1 *rebounds* only if $m_2 > m_1$. If $m_2 = m_1$, however, we find $v_1' = 0$ (object 1 is at rest after the collision) and $v_2' = v_1$.

Note that an explicit solution for the problem of elastic collisions in one dimension was obtained: given the mass ratio $\alpha = m_1/m_2$ and the velocities v_1 and v_2 before the

collision, the velocities v_1' and v_2' could be determined uniquely. This is because we had two unknowns (in one dimension) and two equations (the conservation laws of momentum and kinetic energy).

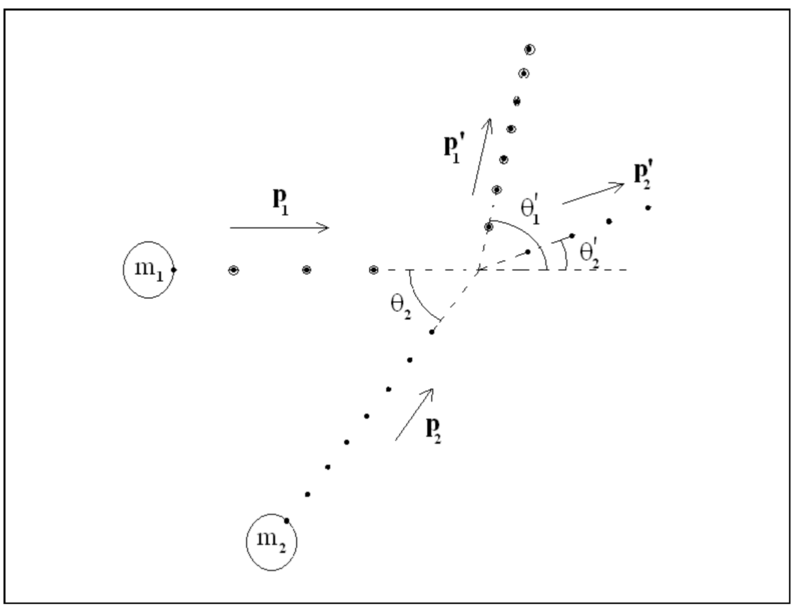
- **Collisions in Two Dimensions**

An explicit solution for the problem of elastic collisions in two dimensions cannot be obtained. Indeed, the four components $\mathbf{v}_1' = (v_{1x}', v_{1y}')$ and $\mathbf{v}_2' = (v_{2x}', v_{2y}')$ cannot be determined from the mass ratio $\alpha = m_1/m_2$ and the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ before the collision since the conservation of momentum and kinetic energy only give us $2+1 = 3$ equations. Hence, one velocity component after the collision must be measured in order for the remaining three components to be determined uniquely from the conditions before the collision.

A simple way to proceed is to assume that the direction of motion one of the two colliding objects (say object 1) define the $\widehat{\mathsf{x}}$-direction, so that the momentum of particle 1 before the collision is simply $\mathbf{p}_1 = p_1\, \widehat{\mathsf{x}}$. The momentum of particle 2 before the collision can now be expressed as

$$\mathbf{p}_2 \;=\; p_2\; (\cos\theta_2\, \widehat{\mathsf{x}} \;+\; \sin\theta_2\, \widehat{\mathsf{y}})\,,$$

where the angle θ_2 is measured from the x-axis as shown in the Figure below.



The momenta of particles 1 and 2 after the collision are also expressed

$$\begin{aligned}
\mathbf{p}_1' &= p_1'\; (\cos\theta_1'\, \widehat{\mathsf{x}} \;+\; \sin\theta_1'\, \widehat{\mathsf{y}}) \\
\mathbf{p}_2' &= p_2'\; (\cos\theta_2'\, \widehat{\mathsf{x}} \;+\; \sin\theta_2'\, \widehat{\mathsf{y}})\,,
\end{aligned}$$

where the angles θ_1' and θ_2' are also measured from the x-axis (see Figure above). Conservation of momentum is now simultaneously applied in the x- and y-directions:

$$p_1 \;+\; p_2\; \cos\theta_2 \;\;=\;\; p_1'\; \cos\theta_1' \;+\; p_2'\; \cos\theta_2'$$

$$p_2 \sin\theta_2 \;=\; p_1' \sin\theta_1' \;+\; p_2' \sin\theta_2'$$

Conservation of kinetic energy, on the other hand, yields

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{{p_1'}^2}{2m_1} + \frac{{p_2'}^2}{2m_2}.$$

Once again, the four unknowns $(p_1', \theta_1';\; p_2', \theta_2')$ cannot be solved uniquely from the initial conditions $(p_1;\; p_2, \theta_2)$ and the mass ratio $\alpha = m_1/m_2$ unless one of the unknowns is measured experimentally (e.g., the deflection angle θ_1').

- **Center of Mass and its Linear Momentum**

An important consequence of the conservation law of momentum in any collision is that the momentum of the **center of mass** is unperturbed by the collision. To prove this statement, we begin with the definition of position $\mathbf{R}_{CM}$ of the center of mass (CM) of two particles

$$\mathbf{R}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.$$

Next, we introduce the velocity $\mathbf{V}_{CM}$ of the center of mass

$$\mathbf{V}_{CM} = \frac{d\mathbf{R}_{CM}}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2},$$

which implies that the momentum of the center of mass, defined as $\mathbf{P}_{CM} = (\sum_i m_i)\,\mathbf{V}_{CM}$, is also expressed as $\mathbf{P}_{CM} = \mathbf{P}$, i.e., the total momentum of a system of colliding particles is equal to the momentum of its center of mass. Consequently, since the total momentum $\mathbf{P}$ is conserved by collisions, the momentum of the center of mass $\mathbf{P}_{CM}$ is also conserved by collisions.

Test your knowledge: Problems 9, 22 & 33 of Chapter 9

11 Rotation Kinematics & Dynamics

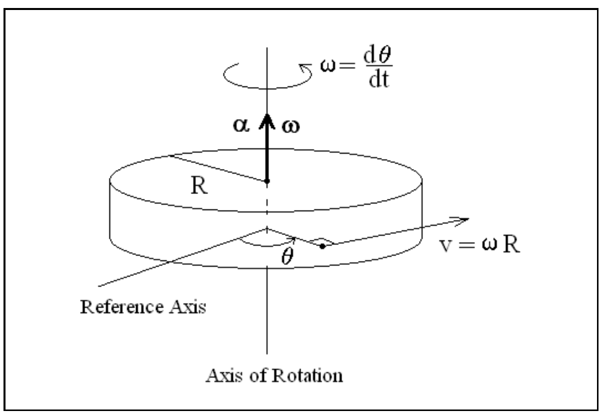
Textbook Reference: Chapter 10 – sections 1-6.

- **Linear and Angular Kinematics**

Up until now, we have considered the kinematics and dynamics of *point* objects, i.e., we considered the motion of the center of mass of objects. We now consider the kinematics and dynamics of *finite* objects (for which the mass of an object is distributed arbitrarily around its center of mass). One new feature associated with this generalization is that, in the course of its motion, an object may rotate about an instantaneous axis of rotation, which may or may not go through the center of mass. The Table below shows the correspondence between linear motion associated with translation and angular motion associated with rotation.

	Linear Kinematics	**Angular Kinematics**
Position	$x(\mathrm{m})$	$\theta(\mathrm{rad})$
Displacement	$\Delta x(\mathrm{m})$	$\Delta\theta(\mathrm{rad})$
Velocity	$v(\mathrm{m/s}) = dx/dt$	$\omega(\mathrm{rad/s}) = d\theta/dt$
Acceleration	$a(\mathrm{m/s^2}) = dv/dt$	$\alpha(\mathrm{rad/s^2}) = d\omega/dt$

Consider, for example, a disk of radius R rotating about an axis of rotation (which goes through its center of mass).



To simplify our analysis, we, henceforth, focus our attention on **rigid bodies**, i.e., bodies for which the distribution of mass is constant. By choosing an arbitrary reference axis from which we measure the instantaneous angular position $\theta(t)$, we can determine the instantaneous angular velocity $\omega(t) = d\theta(t)/dt$; angular-velocity units are either rad/s or rpm (**r**evolutions **p**er **m**inute), where 1 rpm = $(2\pi\,\mathrm{rad})/(60\,\mathrm{s})$. As a result of the disk's rotation, an arbitrary point inside the disk, at a distance r from the axis of rotation, is moving with instantaneous tangential velocity $v(t) = r\,\omega(t)$. Note here that ω denotes the magnitude of the angular-velocity vector $\boldsymbol{\omega}$ directed along the axis of rotation (call it the z-axis) and that the tangential velocity is defined as

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = r\,\frac{d}{dt}(\cos\theta\,\widehat{\mathsf{x}} + \sin\theta\,\widehat{\mathsf{y}}) = r\,\frac{d\theta}{dt}(-\sin\theta\,\widehat{\mathsf{x}} + \cos\theta\,\widehat{\mathsf{y}}) \\ &= (\omega\,\widehat{\mathsf{z}}) \times \mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}.\end{aligned}$$

where we have introduced the **vector** product $\times$ defined for two arbitrary 3D vectors $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} \times \mathbf{B} \;=\; (A_y\,B_z \;-\; A_z\,B_y)\,\widehat{\mathsf{x}} \;+\; (A_z\,B_x \;-\; A_x\,B_z)\,\widehat{\mathsf{y}} \;+\; (A_x\,B_y \;-\; A_y\,B_x)\,\widehat{\mathsf{z}}.$$

Next, the instantaneous acceleration vector experienced by our arbitrary point is now calculated as

$$\mathbf{a} \;=\; \frac{d\mathbf{v}}{dt} \;=\; \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} \;+\; \boldsymbol{\omega}\times\mathbf{v} \;=\; \boldsymbol{\alpha}\times\mathbf{r} \;+\; \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r})\,,$$

where

$$\boldsymbol{\alpha} \;=\; \frac{d\boldsymbol{\omega}}{dt} \;=\; \frac{d\omega}{dt}\,\widehat{\mathsf{z}} \;=\; \alpha\,\widehat{\mathsf{z}}$$

denotes the angular acceleration vector (also directed along the axis of rotation). Note that the acceleration vector has two components: a *tangential* component $\mathbf{a}_{\rm tan}$ and a *radial* component $\mathbf{a}_{\rm r}$. The tangential acceleration $a_{\rm tan} = \alpha\,r$ is produced by the angular acceleration α while the radial acceleration

$$\mathbf{a}_{\rm r} \;=\; \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r}) \;=\; -\,\omega^2\,\mathbf{r}$$

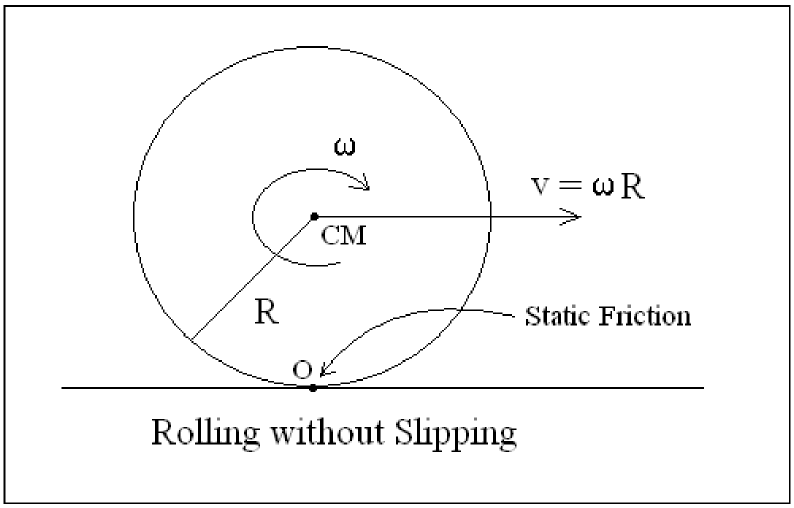
represents the radially-inward *centripetal* acceleration.

- **Kinematic Equations for Uniformly Accelerated Motion**

	Linear Motion	**Angular Motion**
Position	$x(t) = x_0 + v_0\,t + \frac{1}{2}\,a\,t^2$	$\theta(t) = \theta_0 + \omega_0\,t + \frac{1}{2}\,\alpha\,t^2$
Velocity	$v(t) = v_0 + a\,t$	$\omega(t) = \omega_0 + \alpha\,t$
Work-Energy Theorem	$\Delta v^2 = 2\,a\,\Delta x$	$\Delta\omega^2 = 2\,\alpha\,\Delta\theta$

- **Rolling Motion: Translation versus Rotation**

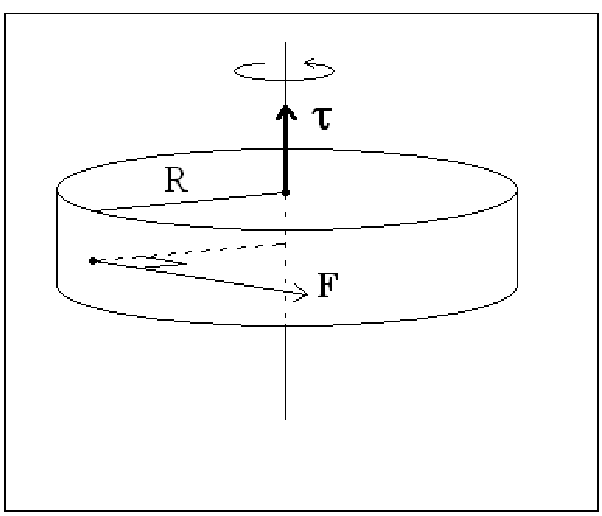
When an object is rolling without slipping on a rough surface (with a finite coefficient of static friction), the translation velocity v of the center of mass (CM) and the angular velocity ω are simply related as $v = \omega R$, where R is the distance between the center of mass and the point of contact O with the surface (see Figure below).



The point of contact O may then be viewed as the *instantaneous* axis of rotation.

- **Torque & Rotational Dynamics**

The physical agent which causes an object to undergo angular acceleration (i.e., an increase in angular velocity) is called **torque**. In simple terms, a torque (denoted as τ) is applied to object when a force **F** is applied at a distance R away from the axis of rotation (see Figure below).



Torque $\boldsymbol{\tau}$ is a vector quantity (directed along the axis of rotation) defined as

$$\boldsymbol{\tau} \;=\; \mathbf{R} \times \mathbf{F},$$

and, thus, the force $\mathbf{F}$ must be at right angle with respect to $\mathbf{R}$. The perpendicular distance from the axis of rotation to the line along which the force acts is called the **lever arm** and the magnitude of the torque vector is written as

$$\tau \;=\; R_{\perp}\, F \;=\; R\,F\, \sin\theta,$$

where θ is the angle betwen the vectors $\mathbf{F}$ and $\mathbf{R}$. Note that the SI unit for torque is $\mathrm{N}\cdot\mathrm{m}$ (not J).

As a simple example, we consider the case of an object of mass m attached by a massless rod of length R to an axis perpendicular to the rod. When a force F is applied to the object (both perpendicular to the rod and the axis), the applied torque $\tau = F\,R$. Since the force F generates a tangential acceleration $a = F/m$, which in turn, is associated with the angular acceleration $\alpha = a/R$, we find

$$\tau \;=\; ma\,R \;=\; \alpha\, mR^2.$$

The *Second Law of Rotational Motion*, therefore, states that the angular acceleration α is directly proportional to the applied torque τ and inversely proportional to the *rotational* inertia mR^2 of the object. This rotational inertia, called moment of inertia and denoted as I, is the analog of mass for translational motion.

Test your knowledge: Problems 4, 20 & 21 of Chapter 10

12 Problems in Rotational Dynamics

Textbook Reference: Chapter 10 – sections 7, 9-11.

- **Moment of Inertia**

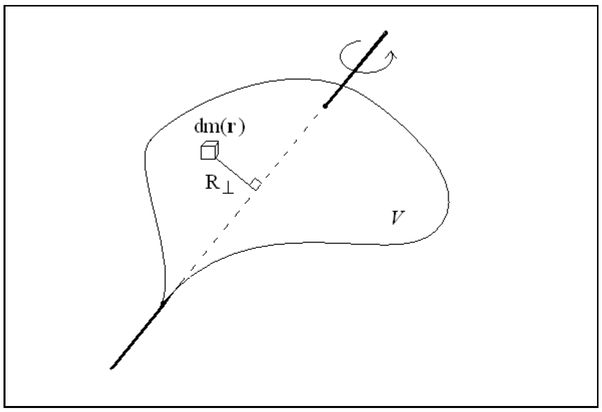
The moment of inertia of a *discrete* distribution of particles $\{(m_1, \mathbf{r}_1), \cdots, (m_N, \mathbf{r}_N)\}$ being rotated about the z-axis is calculated as

$$I \;=\; \sum_{n=1}^{N} m_n \left(x_n^2 \;+\; y_n^2\right).$$

For a general *continous* mass distribution, where an infinitesimal amount of mass $dm(\mathbf{r})$ at point $\mathbf{r}$ is expressed in terms of the mass density $\rho(\mathbf{r})$ and the infinitesimal volume element $d\tau$ as $dm(\mathbf{r}) = \rho(\mathbf{r})\, d\tau$, the moment of inertia of an object rotated about an arbitrary axis is

$$I \;=\; \int R_{\perp}^2 \; dm(\mathbf{r}) \;=\; \int_V \rho(\mathbf{r})\, R_{\perp}^2(\mathbf{r})\; d\tau,$$

where $R_{\perp}(\mathbf{r})$ is the perpendicular distance from point $\mathbf{r}$ to the axis of rotation (see Figure below).



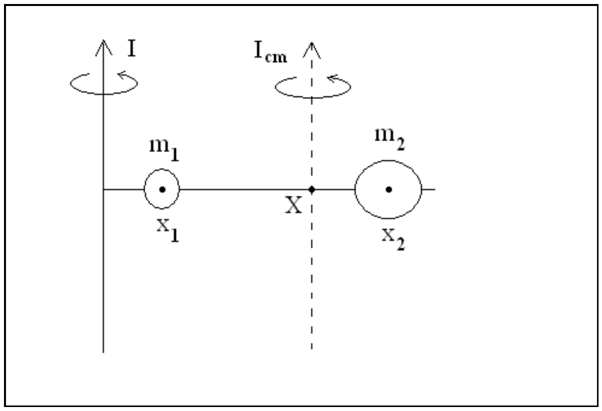
For example, the moment of inertia of a uniform disk of mass m, radius R, and thickness h being rotated about its axis of symmetry (going through its center of mass) is

$$I_{cm} \;=\; \int_0^R r\, dr \int_0^{2\pi} d\theta \int_0^h dz \left(\frac{m}{\pi\, R^2 h} \right) r^2 \;=\; \frac{1}{2}\, mR^2,$$

where $\rho = m/(\pi\, R^2 h)$ denotes the uniform density of the disk.

Note that the moment of inertia depends on the distribution of mass about the axis of rotation and, consequently, an object will have a different moment of inertia if the axis of rotation is moved from one place to another. For example, we consider the moment of inertia of a system composed of two particles of mass m_1 and m_2 located on the x-axis at x_1 and x_2 respectively. Let us first calculate the moment of inertia about the z-axis (i.e., about $x = 0$):

$$I \;=\; m_1\, x_1^2 \;+\; m_2\, x_2^2.$$



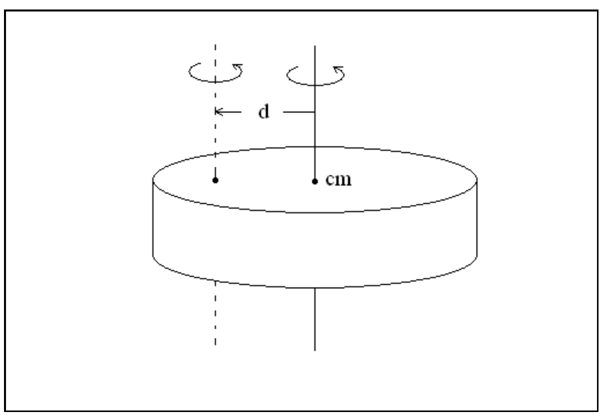
Next, we move the axis of rotation to the position of the center of mass (see Figure above)

$$X \;=\; \frac{m_1}{M}\,x_1 \;+\; \frac{m_2}{M}\,x_2,$$

where $M = m_1 + m_2$ denotes the total mass of the system, so that the moment of inertia is now

$$I_{cm} \;=\; m_1\;(x_1 \;-\; X)^2 \;+\; m_2\;(x_2 \;-\; X)^2 \;=\; I \;-\; M\,X^2.$$

This result leads to the **Parallel-Axis Theorem**: $I_d = I_{cm} + M\,d^2$. Hence, if we move the axis of rotation in the case of the disk discussed above by a distance d from the center of mass (see Figure below),



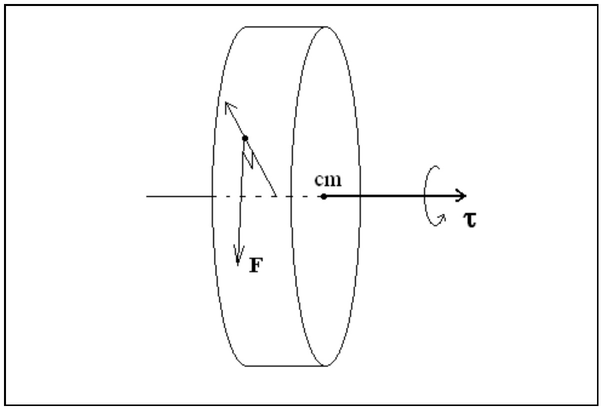
we find the new moment of inertia

$$I_d \;=\; I_{cm} \;+\; m\,d^2 \;=\; m\left(\frac{1}{2}\,R^2 \;+\; d^2\right),$$

where the first contribution comes from the moment of inertia as calculated from the center of mass and the second contribution comes from the displacement of the axis of rotation.

- **Rotational Dynamics**

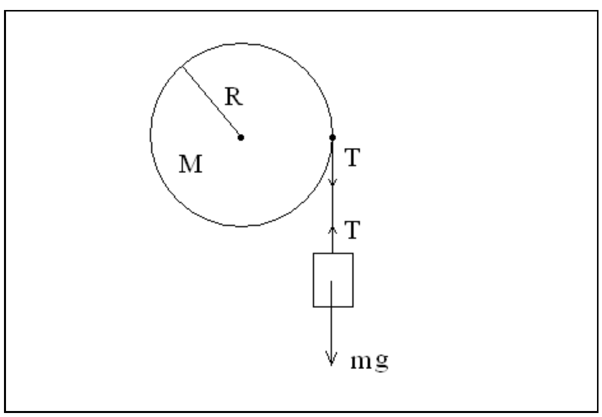
We now consider what happens when a force is applied tangentially to a uniform disk of mass m, radius R, and thickness h, which is allowed to rotate about an axis passing through its center of mass (see Figure below).



The torque applied on the disk is $\tau = F\,R$ (since the force is applied at a distance R from the axis of rotation), and the Second Law of Rotational Dynamics states that this net torque causes the disk to acquire angular acceleration

$$\alpha \;=\; \frac{\tau}{I} \;=\; \frac{F\,R}{\frac{1}{2}\,mR^2} \;=\; \frac{2\,F}{mR}.$$

As an application, consider the *half*-Atwood machine shown in the Figure below



in which an object of mass m is attached to a (massless) string wrapped (tightly) around a disk pulley of mass M and radius R. As a result of the downward gravitational force on the object, the force analysis on the object shows that its downward acceleration is

$$m\,a \;=\; m\,g \;-\; T,$$

where T denotes the tension in the string. The tension in the string, in turn, exerts a torque $\tau = T\,R$, which causes the pulley to rotate with angular acceleration

$$\alpha \;=\; \frac{2\,T}{MR}.$$

Since the string is wrapped tightly around the pulley, there is no slipping and the acceleration a and the angular acceleration α are related: $a = \alpha\,R$ and, thus, we find the tension in the string as $T = \frac{1}{2}\,Ma$. By combining these two results, we find the system acceleration and string tension

$$a \;=\; \left(\frac{m}{m + M/2} \right) g \quad \text{and} \quad T \;=\; \left(\frac{m\,(M/2)}{m + M/2} \right) g,$$

respectively. As expected, if $M = 0$ (i.e., for a massless pulley), we find $a = g$ and $T = 0$.

- **Angular Momentum and its Conservation**

In analogy with linear momentum $\mathbf{p} = m\mathbf{v}$, we define the angular momentum $\mathbf{L} = I\,\boldsymbol{\omega}$ as the product of the moment of inertia of an object being rotated about a fixed axis of rotation with angular velocity $\boldsymbol{\omega}$ (directed along that axis). The Second Law of Rotational Dynamics can, therefore, be expressed as

$$\boldsymbol{\tau} \;=\; \frac{d\mathbf{L}}{dt},$$

i.e., the rate of change of the angular momentum defines torque. If the external torque on an object is zero, the angular momentum of that object is conserved. Hence, by changing the moment of inertia of an object $I \rightarrow I'$, we may change the angular velocity

$$\omega \;\rightarrow\; \omega' \;=\; \left(\frac{I}{I'} \right) \omega.$$

- **Rotational Kinetic Energy**

Once again, proceeding by analogy with the linear case, the kinetic energy associated with rotation is defined as

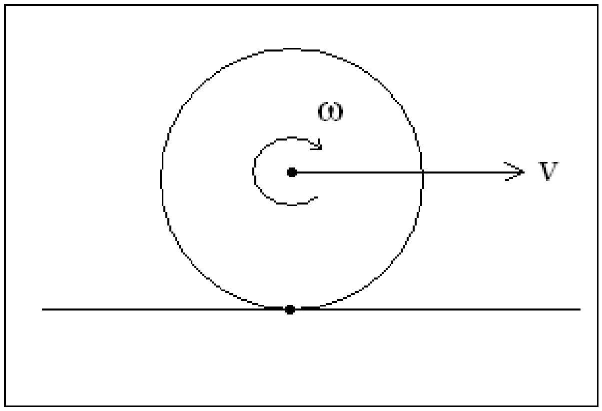
$$K_{rot} \;=\; \frac{1}{2}\,I\,\omega^2.$$

The Work-Energy Theorem associated with rotational dynamics is, therefore, expressed as

$$\Delta K_{rot} \;=\; W_{net} \;=\; \int \tau\; d\theta.$$

- **Rolling without Slipping**

We combine the two forms of motion (linear and angular) and consider the motion of an object of mass m and radius R **rolling without slipping** on a horizontal surface (see Figure below).



The "no-slip" condition implies that the linear velocity v (with which the center of mass travels) is related to the angular velocity ω: $v = \omega\, R$. The total kinetic energy is, therefore, a sum of its linear and angular parts

$$K \;=\; \frac{1}{2}\, mv^2 \;+\; \frac{1}{2}\, I\,\omega^2 \;=\; \left(1 \;+\; \frac{I}{m\,R^2}\right) \frac{1}{2}\, m\,v^2 \;=\; (1+k)\; K_{lin},$$

where $k = I/(mR^2)$ is a dimensionless number (e.g., $k = \frac{1}{2}$ for a disk, $k = \frac{2}{5}$ for a sphere, and $k = 1$ for a hoop). Hence, if we apply the Energy Conservation Law $\Delta K + \Delta U = 0$ to the case where this object rolls down an incline (without slipping) and calculate how fast the object is moving after having dropped by an effective height h, we find

$$\Delta K \;=\; \frac{(1+k)}{2}\, m\,v^2 \;=\; -\,\Delta U \;=\; mgh \quad\rightarrow\quad v \;=\; \sqrt{\frac{2gh}{1+k}}.$$

The fastest object down the incline is, therefore, the one which does NOT rotate (i.e., a box $\rightarrow k = 0$), while the slowest object is the one for which $k = 1$ (i.e., a hoop).

Test your knowledge: Problems 42, 52 & 61 of Chapter 10

13 Statics

Textbook Reference: Chapter 12 – sections 1-3.

- **Force and Torque Equilibria**

An object subjected to forces $(\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_N)$, applied at different locations on the object, is in **static equilibrium** if the net force $\sum_{n=1}^{N} \mathbf{F}_n$ and the net torque $\sum_{n=1}^{N} \boldsymbol{\tau}_n$ on the object both vanish. Assuming that the forces are planar (say in the x-y plane), then the conditions of static equilibrium become

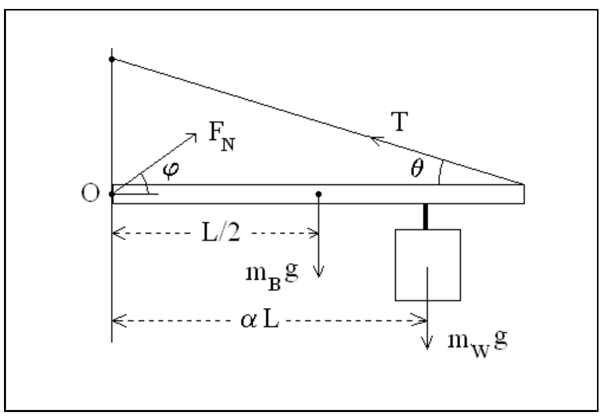
$$\sum_{n=1}^{N} F_{n\,x} \;=\; 0 \;=\; \sum_{n=1}^{N} F_{n\,y}, \tag{12}$$

$$\sum_{n=1}^{N} \tau_{n\,z} \;=\; 0, \tag{13}$$

where the torque produced by an (x, y)-planar force is directed along the z-axis.

- **Statics Problems**

As an example, we consider the case of a uniform beam (B) of length L and mass m_B attached on a wall at point O (see Figure below) with the help of a cable (assumed massless) so that the beam is hanging horizontally while the cable makes an angle θ with respect to the horizontal. Next, attached to the beam at a distance $\alpha\, L$ from the wall, we place a weight (W) of mass m_W.



This problem requires that the tension T in the cable and the normal force

$$\mathbf{F}_N \;=\; F_N\;(\cos\varphi\,\widehat{\mathsf{x}} \;+\; \sin\varphi\,\widehat{\mathsf{y}})$$

provided by the wall be calculated by using the three static-equilibrium conditions (12) and (13). First, the condition for force equilibrium in the x-direction requires that

$$F_N\;\cos\varphi \;=\; T\;\cos\theta,$$

while the condition for force equilibrium in the y-direction requires that

$$F_N\;\sin\varphi \;+\; T\;\sin\theta \;=\; (m_B \;+\; m_W)\;g.$$

Next, the condition for torque equilibrium requires that

$$TL\,\sin\theta \;=\; m_B g\,\frac{L}{2} \;+\; m_W g\,\alpha\,L,$$

where the torque exerted by the weight of the beam is calculated by placing the full mass of the beam at its center of mass located at its center (at a distance $L/2$ from point O on the wall).

From the last equation, we find the tension

$$T \;=\; \left(\frac{1}{2}\,m_B \;+\; \alpha\,m_W\right)\,\frac{g}{\sin\theta},$$

while the amplitude of the normal force F_N and its angle φ are now determined from the two equations

$$\begin{aligned} F_N\,\cos\varphi &= \left(\frac{1}{2}\,m_B \;+\; \alpha\,m_W\right)\,g\,\cot\theta, \\ F_N\,\sin\varphi &= \left[\frac{1}{2}\,m_B \;+\; (1-\alpha)\,m_W\right]\,g. \end{aligned}$$

From these equations, we solve for the angle φ as

$$\tan\varphi \;=\; \tan\theta\;\cdot\;\left[\frac{m_B + 2\,(1-\alpha)\,m_W}{m_B + 2\alpha\,m_W}\right],$$

while the magnitude F_N is solved as

$$F_N \;=\; g\,\sqrt{\left[\frac{m_B}{2} + (1-\alpha)\,m_W\right]^2 \;+\; \left[\frac{m_B}{2} + \alpha\,m_W\right]^2\,\cot^2\theta}.$$

Note that if $m_W = 0$, we find $\varphi = \theta$ and $F_N = T = m_B\,g/(2\,\sin\theta)$.

Test your knowledge: Problems 3 & 5 of Chapter 10

14 Oscillations

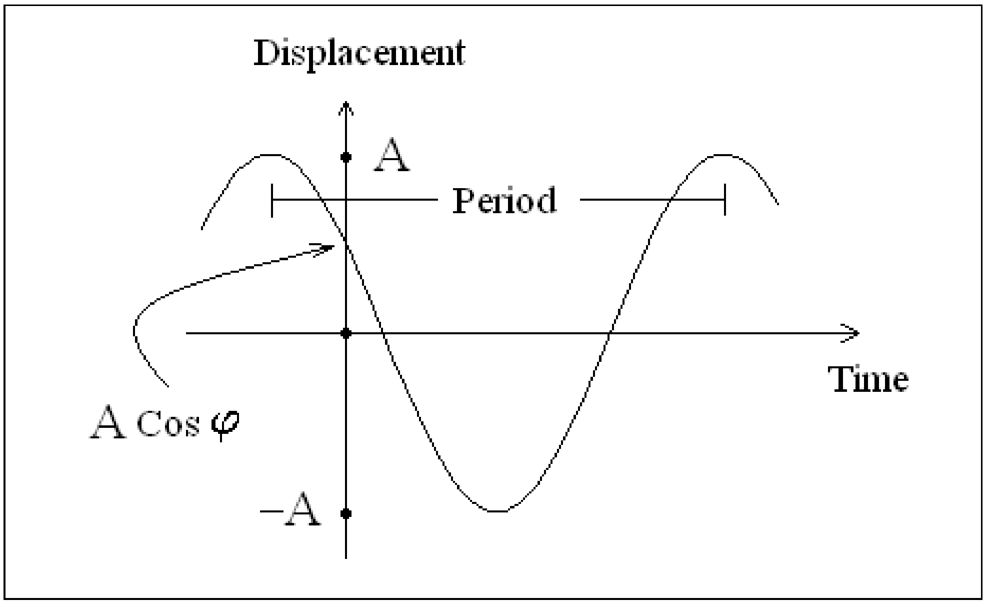
Textbook Reference: Chapter 14 – sections 1-8.

- **Simple Harmonic Motion of a Mass on a Spring**

The equation of motion for a mass m is attached to a spring of constant k is

$$m\,a \;=\; m\,\frac{d^2x}{dt^2} \;=\; -\,k\,x,$$

where the restoring force of the spring, $F = -\,k\,x$, is linearly proportional to the spring displacement x from away from its equilibrium state.



Using the initial conditions $x(t=0) = x_0$ and $v(t=0) = v_0$, we find that the solution $x(t)$ for this motion is an oscillatory function of time (see Figure above)

$$x(t) \;=\; A\,\cos(\omega\,t \;+\; \varphi), \tag{14}$$

where A denotes the amplitude of the oscillation, φ is the initial phase of the oscillatory motion so that

$$x_0 \;=\; A\,\cos\varphi \quad \text{and} \quad v_0 \;=\; -\,A\,\omega\,\sin\varphi \;=\; -\,\omega\,x_0\,\tan\varphi,$$

and

$$\omega \;=\; 2\pi\,f \;=\; \frac{2\pi}{T} \;=\; \sqrt{\frac{k}{m}}$$

denotes the angular *frequency* (with units rad/s), while $f = 1/T$ denotes the frequency (with units Hz $= \mathrm{s}^{-1}$) defined as the inverse of the period T.

Since the restoring force is linearly proportional to the displacement x, the oscillatory motion (14) is called *Simple Harmonic Motion* (SHM) and the mass-spring system is known as a *simple harmonic oscillator* (SHO).

- **Energy of a Simple Harmonic Oscillator**

The total energy for a mass-spring system

$$E \;=\; \frac{m}{2}\,v^2 \;+\; \frac{k}{2}\,x^2$$

can be expressed in terms of the SHM solution (14) by, first, writing the expression for the velocity

$$v(t) \;=\; \frac{dx}{dt} \;=\; -\,A\,\omega\;\sin(\omega\,t \;+\; \varphi).$$

If we now substitute this expression into the energy expression, we find

$$E \;=\; \frac{A^2}{2}\left[\,m\,\omega^2\;\sin^2(\omega\,t \;+\; \varphi) \;+\; k\;\cos^2(\omega\,t \;+\; \varphi)\,\right] \;=\; \frac{1}{2}\left\{\begin{array}{l} k\,A^2 \\ \\ \text{or} \\ \\ m\,\omega^2\,A^2 \end{array}\right.$$

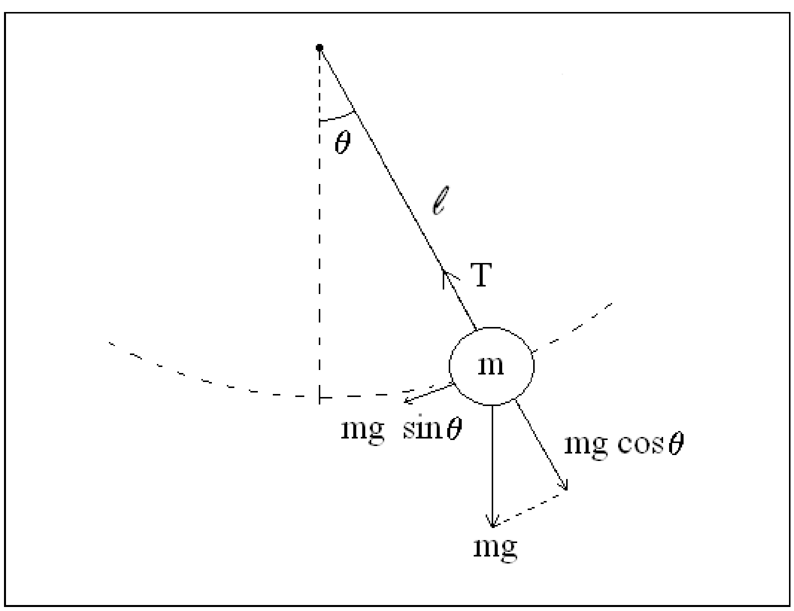
where we have used the definition $\omega = \sqrt{k/m}$ for the angular frequency. Hence, we find that the energy is indeed a constant of the motion (it is determined solely from the initial conditions) and is directly proportional to the **square** of the amplitude A (i.e., by doubling the amplitude of an oscillation, we quadruple its energy).

- **Simple Pendulum**

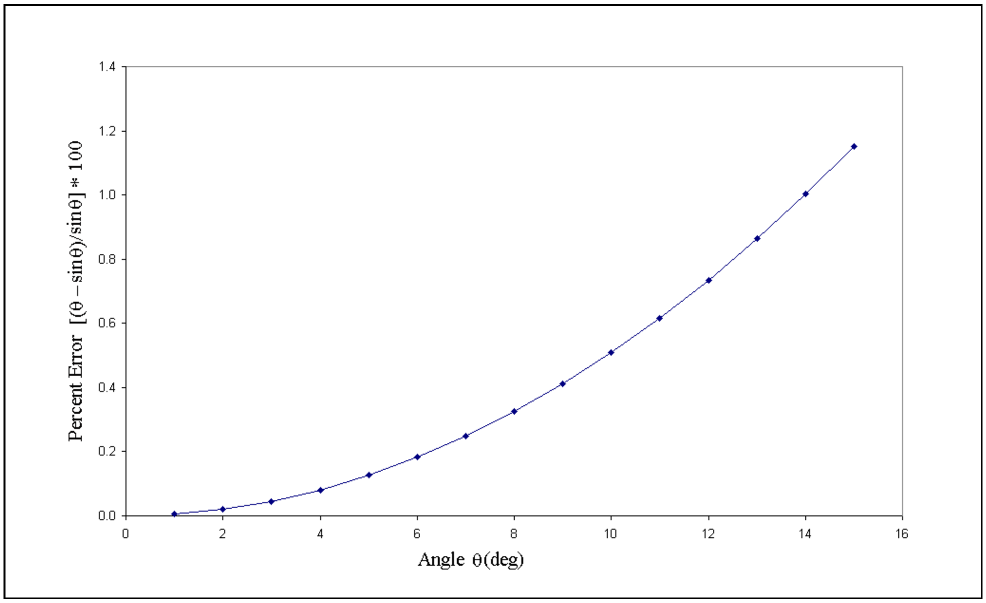
The equation of motion for a pendulum of mass m and length ℓ is expressed as

$$m\ell\,\frac{d^2\theta}{dt^2} \;=\; -\,mg\;\sin\theta, \tag{15}$$

where θ denotes the angular displacement of the pendulum away from the vertical (equilibrium) line (see Figure below).



If θ remains below approximately 15^o, we may replace $\sin\theta$ with θ (where θ MUST be expressed in radians); the graph below shows the percent error $[(\theta - \sin\theta)/\sin\theta] \times 100$ as a function of θ (here expressed in degrees), and we clearly see that this *simple-pendulum* approximation has better than 1% accuracy for angles below $\sim 15^o$.



Under the simple-pendulum approximation ($\sin\theta \simeq \theta$), the equation of motion (15), therefore, becomes

$$m\ell \, \frac{d^2\theta}{dt^2} \; = \; - \, mg \, \theta, \tag{16}$$

whose solution is of the form

$$\theta(t) \; = \; \Theta \; \cos(\omega \, t \; + \; \varphi),$$

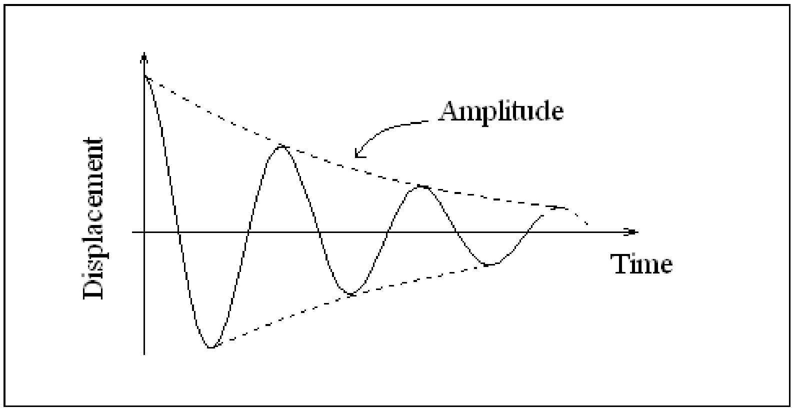
where Θ and φ denote the amplitude and initial phase of the simple harmonic motion of the simple pendulum and the angular frequency ω of the simple pendulum is

$$\omega \; = \; \sqrt{\frac{g}{\ell}}.$$

This result could have been obtained on simple dimensional grounds and it is interesting to note that the mass of the pendulum does not enter since mass is represents both inertial effects and the restoring force and, thus, it cancels out.

• Damped Oscillations

When a simple harmonic oscillator is exposed to dissipation, the amplitude of oscillations decreases as a function of time as the oscillation energy goes to zero (see Figure below).



As a simple model to investigate *damped* oscillations, we consider the case of an object of mass m attached to a spring of constant k and exposed to a dissipative force of the form $F_{diss} = -m\,\nu\,v$, where ν denotes an energy-dissipation rate and $v = dx/dt$ denotes the instantaneous velocity of the block. The equation of motion becomes

$$\frac{d^2x}{dt^2} \;+\; \nu\,\frac{dx}{dt} \;+\; \omega^2\,x \;=\; 0,$$

where $\omega = \sqrt{k/m}$ denotes the undamped angular frequency. Solutions of this second-order differential equation are generally studied in a course on differential equations; in the interest of expediency, we simply introduce the following solution in the form

$$x(t) \;=\; A\,e^{-\nu\,t/2}\,\cos(\Omega\,t \;+\; \varphi),$$

where A denotes the initial amplitude of the damped oscillation and the modified angular frequency

$$\Omega \;=\; \sqrt{\omega^2 \;-\; \frac{\nu^2}{4}}$$

depends on the properties of the undamped system as well as the energy-dissipation rate.

Note that, for this solution, we can consider three cases. In case I, the *underdamped* case, the energy-dissipation ν is small (i.e., $\nu < 2\,\omega$), and the system performs several oscillations before the amplitude decreases significantly. In case II, the *overdamped* case, the energy-dissipation ν is large (i.e., $\nu > 2\,\omega$), and the system cannot even undergo a single oscillation cycle (i.e., the solution is purely exponentially decreasing since $\Omega^2 < 0$). In case III, the *critical-damping* case, the energy-dissipation rate $\nu = 2\,\omega$ is such that $\Omega = 0$ (i.e., the motion is purely exponentially decreasing, just as in case II). In all three cases, the equilibrium state is reached exponentially, with case III exhibiting the fastest approach.

Lastly, we note that the rate with which the mechanical energy $E = \frac{1}{2}(m\,v^2 + k\,x^2)$:

$$\frac{dE}{dt} \;=\; \left(m\,\frac{d^2x}{dt^2} \;+\; k\,x\right)\frac{dx}{dt} \;=\; -\,2\,\nu\,K \;<\; 0$$

is both proportional to the energy-dissipation rate ν as well as the kinetic energy $K = \frac{1}{2}\,mv^2$ of the block (i.e., energy dissipation is strongest where kinetic energy is largest).

Test your knowledge: Problems 3, 27 & 39 of Chapter 14

15 Waves

Textbook Reference: Chapter 15 – sections 1-4 & 6-9.

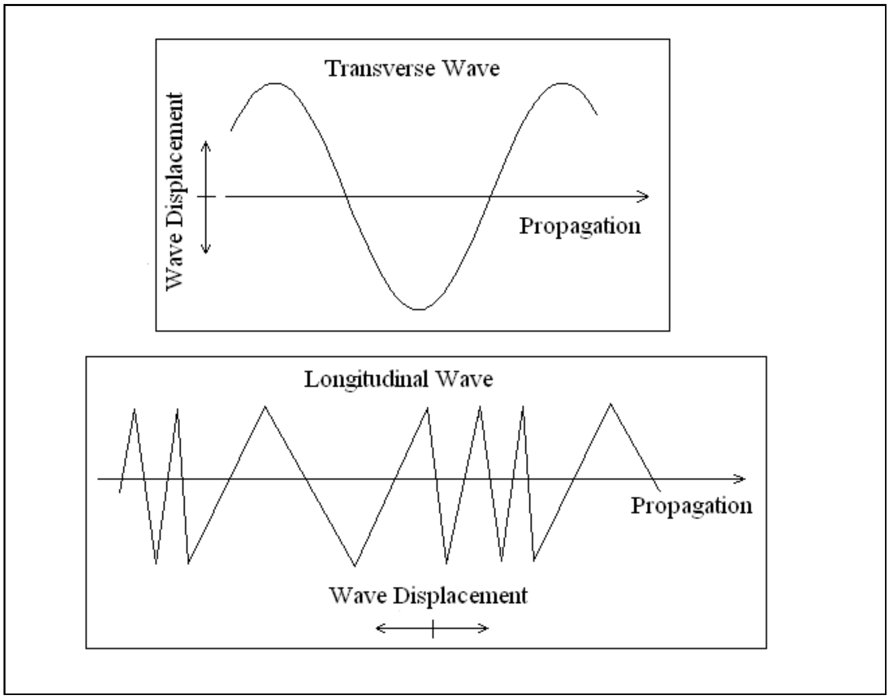
- **Characteristics of Wave Motion**

Like an oscillation, a wave has an amplitude D, a frequency $f = \omega/(2\pi)$, and a period $T = 1/f$. In addition, however, since a wave is periodic in **both** space and time, a wave has a wavelength $\lambda = 2\pi/k$ (where k is called the wavenumber) and it travels with wave speed

$$v \;=\; \lambda\, f \;=\; \frac{\lambda}{T} \;=\; \frac{\omega}{k}.$$

Note that waves appear either as continuous waves, as wave packets, or as pulses. In the latter case, the pulse can be described in terms of an amplitude and a pulse speed (which is identical to its wave speed if the pulse were a continuous wave).

Waves can be divided into two different types: transverse waves or longitudinal waves. A transverse wave has its wave-displacement axis perpendicular to its propagation axis while a longitudinal wave has its wave-displacement axis parallel to its propagation axis (see Figure below).



Examples of transverse waves include waves at the surface of water and light waves (with minima and maxima referred to as throughs and crests, respecitvely), while examples of longitudinal waves include sound waves (with minima and maxima referred to as compressions and expansions, respectively).

Wave speeds for transverse and longitudinal waves depend on properties of the medium in which they propagate. Since wave speed depends on the frequency f of the wave, then the wave speed depends on the restoring-force and inertial properties of the medium. For a transverse wave travelling on a stretched string, for example, the restoring force is provided by the tension F_T(N) in the string while inertia is represented by the linear mass density μ(kg/m) of the string. From simple dimensional analysis, we find that the wave speed for a transverse wave on a stretched string is $v = \sqrt{F_T/\mu}$, as might be expected on physical grounds. For a longitudinal wave such a sound travelling in a solid, on the other hand, the restoring force is provided by the elastic *modulus* $E(\mathrm{N/m^2})$ of the material and inertia is represented by the mass density $\rho(\mathrm{kg/m^3})$. Once again from simple dimensional analysis, the wave speed of a sound wave travelling in a solid is $v = \sqrt{E/\rho}$.

- **Energy Transported by a Wave**

By simple analogy with oscillations ($E = \frac{1}{2} m\omega^2 \ D^2$), travelling waves also possess energy and because they travel, this wave energy can be transported through space. Indeed, by expressing mass m as

$$m \;=\; \rho\, V \;=\; \rho\ (A\,\ell) \;=\; \rho\, A\, v\, t,$$

which denotes the amount of mass transported across an area A in time t, where ρ is the mass density and A is the cross-sectional area through which the wave travels. Hence, the average rate $\overline{P} = E/t$ at which energy is transported is defined as

$$\overline{P} \;=\; \frac{1}{2}(\rho\; A\; v)\; \omega^2\; D^2 \;=\; 2\pi^2(\rho\; A\; v)\; f^2\; D^2.$$

Lastly, we define the **intensity** I of the wave as the average power transported by the wave across unit area transverse to its propagation axis:

$$I \;=\; \frac{\overline{P}}{A} \;=\; 2\pi^2(\rho\; v)\; f^2\; D^2,$$

i.e., the intensity of a wave depends on the square of its amplitude D and its frequency f and is linearly proportional to its wave speed.

We now note that, for a continuous wave, the rate of wave generation is constant and is proportional to the average wave power $\overline{P}$ and, thus, as the wave propagates outward away from its source, it is spread over a progressively larger area A and, therefore, the wave intensity is inversely proportional to the area A. For a wave produced by a point source and travelling in three dimensions, the area $A = 4\pi\, r^2$ is the area of the surface of a sphere of radius r (note that $v = dr/dt$) and, thus the intensity of a spherical wave decreases with the inverse square of the distance to the source:

$$I \;\propto\; \frac{1}{r^2} \quad\rightarrow\quad \frac{I_1}{I_2} \;=\; \left(\frac{r_2}{r_1}\right)^2.$$

Associated with a decrease in wave intensity, the wave amplitude also decreases with distance to the source (i.e., $A \propto 1/r$ for a spherical wave).

- **Solution for a Travelling Wave**

The mathematical description of a travelling wave involves the function $D(x,t)$ representing the wave displacement at location x at time t defined as

$$D(x,t) \;=\; D_0\; \sin\left(\frac{2\pi\; x}{\lambda} \;\pm\; \frac{2\pi\, t}{T}\right) \;=\; D_0\; \sin(kx \pm \omega t),$$

where the (+)-sign refers to a wave travelling to the left (towards negative x-values) and the $(-)$-sign refers to a wave travelling to the right (towards positive x-values). Note that, if we introduce the wave phase $\Phi(x,t) = k\,x - \omega\,t$, we find that the condition of constant phase

$$\frac{d\Phi}{dt} \;=\; k\,\frac{dx}{dt} \;\pm\; \omega \;=\; 0 \quad\rightarrow\quad v \;=\; \frac{dx}{dt} \;=\; \mp\,\frac{\omega}{k},$$

shows that the wave speed (also called phase speed) is the speed with which a fixed point on the wave is moving (e.g., think of a surfer *riding* the wave).

- **Wave Properties**

All waves have a certain set of properties they have in common; a more detailed account of wave properties (e.g., diffraction) will be presented next Semester in the context of light waves. First, all waves can be reflected at boundaries and can be transmitted from one medium to another medium while undergoing refraction. Next, all waves can experience constructive and destructive interference, which is analysed through the Principle of Superposition. As an example of the subtle interplay of reflection and interference effects, we mention the resonant interference involving counter-propagating waves leading to the formation of standing waves.

Test your knowledge: Problems 1, 3, & 36 of Chapter 15